# Do-calculus

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# Recap about causal graphical models (1/1)

Active and blocked paths A path is said to be blocked by a set of vertices $\mathcal{Z} \in \mathcal{V}$ if:

- it contains a chain $A \rightarrow B \rightarrow C$ or a fork $A \leftarrow B \rightarrow C$ and $B \in \mathcal{Z}$, or
- it contains a collider $A \rightarrow B \leftarrow C$ such that no descendant of $B$ is in $\mathcal{Z}$.

d-separation Given disjoint sets $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{V}$, we say that $\mathcal{X}$ and $\mathcal{Y}$ are d-separated by $\mathcal{Z}$ if every path between a node in $\mathcal{X}$ and a node in $\mathcal{Y}$ is blocked by $\mathcal{Z}$ and we write $\mathcal{X} \perp\!\!\!\perp_G \mathcal{Y} | \mathcal{Z}$.

# Recap about causal graphical models (2/2)

The do() operator allows to represent interventions in equations.

# Recap about the Back-door and Front-door criteria (1/3)

The back-door criterion: Consider a causal graph $\mathcal{G}$ and a causal effect $P(y \mid do(x))$. A set of variables $\mathcal{Z}$ satisfies the back-door criterion iff:

- no node in $\mathcal{Z}$ is a descendant of $X$;
- $\mathcal{Z}$ blocks every path between $X$ and $Y$ that contains an arrow into $X$.

Theorem (back-door adjustment): If $\mathcal{Z}$ satisfies the back-door criterion relative to $(X, Y)$ and if $\Pr(x, z) > 0$, then the causal effect of $X$ on $Y$ is identifiable and is given by

$$\Pr(y \mid do(x)) = \sum_{z} \Pr(y \mid x, z) \Pr(z).$$

# Recap about the Back-door and Front-door criteria (2/3)

Front-door criterion: Consider a causal graph $\mathcal{G}$ and a causal effect $\Pr(y \mid do(x))$. A set of variables $\mathcal{Z}$ satisfies the front-door criterion iff:

- $\mathcal{Z}$ intercepts all directed paths from $X$ to $Y$;
- There is no back-door path from $X$ to $\mathcal{Z}$;
- All back-door paths from $\mathcal{Z}$ to $Y$ are blocked by $X$.

Theorem (front-door adjustment): if $\mathcal{Z}$ satisfies the front-door criterion relative to $(X, Y)$ and if $\Pr(x, z) > 0$, then the causal effect of $X$ on $Y$ is identifiable and is given by

$$\Pr(y \mid do(X = x)) = \sum_{z} \Pr(z \mid x) \sum_{x'} \Pr(y \mid x', z) \Pr(x').$$

# Recap about the Back-door and Front-door criteria (3/3)

- If there exists a set that satisfy the back-door criterion for $\Pr(y \mid do(x))$, then $\Pr(y \mid do(x))$ is identifiable;
- If there exists a no set that satisfy the back-door criterion for $\Pr(y \mid do(x))$, then $\Pr(y \mid do(x))$ is not necesarly not identifiable.
- If there exists a set that satisfy the front-door criterion for $\Pr(y \mid do(x))$, then $\Pr(y \mid do(x))$ is identifiable;
- If there exists a no set that satisfy the fack-door criterion for $\Pr(y \mid do(x))$, then $\Pr(y \mid do(x))$ is not necesarly not identifiable.

The combination of the back-door and front door criteria are also incomplete.

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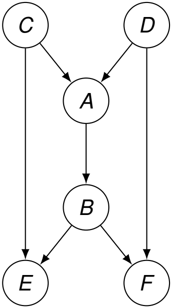
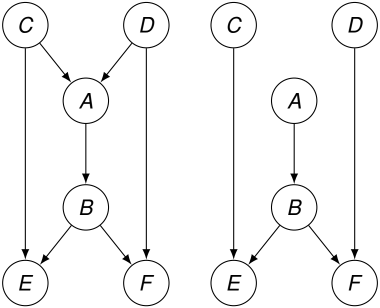
# Mutilated Graphs




$\mathcal{G}$

# Mutilated Graphs




$\mathcal{G}$ $\mathcal{G}_{\overline{A}}$

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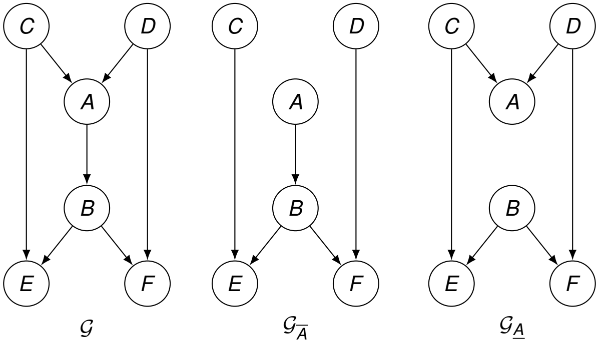

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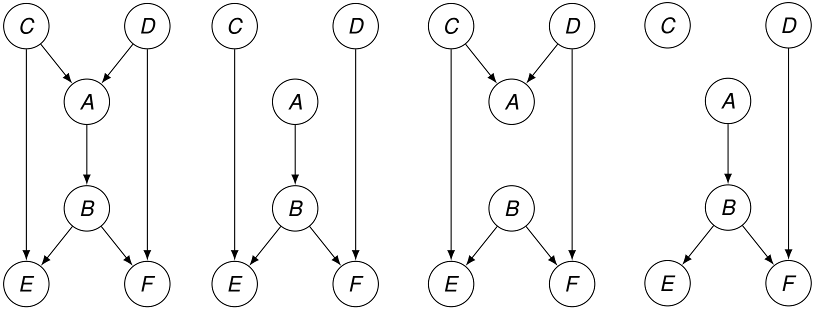


$\mathcal{G}$ $\quad$ $\mathcal{G}_{\overline{A}}$ $\quad$ $\mathcal{G}_{\underline{A}}$ $\quad$ $\mathcal{G}_{\overline{A}\underline{C}}$

# Augmented Graphs

Consider $\Pr(y \mid do(z))$ and the Probabilistic Causal Model:

$$M = < \mathcal{U}, \mathcal{V}, \mathcal{F}, P(\mathcal{U}) >$$

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Augmented model of $M$ for $do(z)$

$$Aug(M, \mathcal{Z}) = < \mathcal{U}, \mathcal{V} \cup \hat{\mathcal{Z}}, \mathcal{F}_{\hat{\mathcal{Z}}}, P(\mathcal{U}) >$$

where $\forall \hat{Z} \in \hat{\mathcal{Z}}$, $\hat{Z}$ represents $do(z)$.

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Augmented graph of $\mathcal{G}$ for $do(z)$

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# Augmented Graphs

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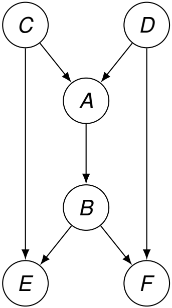
Augmented graph of $\mathcal{G}$ for $do(z)$

$$Aug(\mathcal{G}, \mathcal{Z}) = \mathcal{G} \cup \{\hat{Z} \to Z \mid \forall \hat{Z} \in \hat{\mathcal{Z}}\}$$

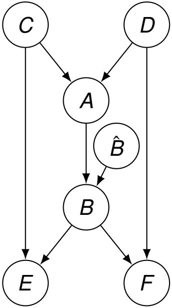
and in the compatible distribution, $\forall Z \in \mathcal{Z}$

$$P(z \mid Pa(z), \hat{z}) = \begin{cases} P(z \mid Pa(z)) & \text{if } \hat{Z} = idle \\ \hat{z} & \text{if } \hat{Z} = do(z) \end{cases}$$

# Example of an augmented graph

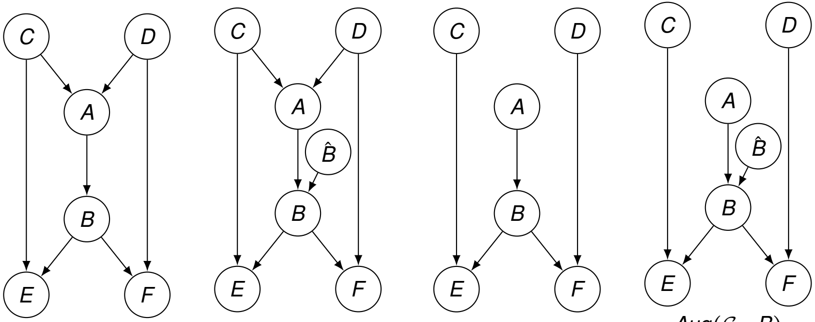



$\mathcal{G}$




$Aug(\mathcal{G}, B)$

# Example of an augmented graph




$\mathcal{G}$ $Aug(\mathcal{G}, B)$ $\mathcal{G}_{\overline{A}}$ $Aug(\mathcal{G}_{\overline{A}}, B)$

# Rule 1: Insertion / deletion of observations

Theorem Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a causal graph. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W} \subseteq \mathcal{V}$ be disjoint. We have:

$$\Pr(y|do(x), z, w) = \Pr(y|do(x), w) \quad \text{if} \quad (\mathcal{Y} \perp\!\!\!\perp \mathcal{Z}|\mathcal{X}, \mathcal{W})_{\mathcal{G}_{\overline{\mathcal{X}}}}$$

(proof on board)

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**Remark:** This Rule is a generalization of d-separation.

# Rule 2: Action/observation exchange

Theorem Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a causal graph. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W} \subseteq \mathcal{V}$ be disjoint. We have:

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Proof: Follows the following Lemma

Lemma Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a causal graph. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W} \subseteq \mathcal{V}$ be disjoint.

$$(\mathcal{Y} \perp\!\!\!\perp \mathcal{Z} \mid \mathcal{X}, \mathcal{W})_{\mathcal{G}_{\overline{\mathcal{X}}\underline{\mathcal{Z}}}} \iff (\hat{\mathcal{Z}} \perp\!\!\!\perp \mathcal{Y} \mid \mathcal{X}, \mathcal{Z}, \mathcal{W})_{Aug(\mathcal{G}_{\overline{\mathcal{X}}\underline{\mathcal{Z}}}, \mathcal{Z})}$$

Proof in (Pearl, 1995)

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Proof in (Pearl, 1995)

**Remark:** This Rule is a generalization of the back-door criterion.

# Rule 3: insertion / deletion of actions

**Theorem** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a causal graph. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W} \subseteq V$ be disjoint. We have:

$$\Pr(y|do(x), do(z), w) = \Pr(y|do(x), w) \quad \text{if} \quad (\mathcal{Y} \perp\!\!\!\perp \mathcal{Z}|\mathcal{X}, \mathcal{W})_{\mathcal{G}_{\overline{\mathcal{X}}\,\overline{\mathcal{Z}(\mathcal{W})}}}$$

where $\mathcal{Z}(\mathcal{W})$ is the set of $\mathcal{Z}$-vertices that are not ancestors of any $\mathcal{W}$-vertex in $\mathcal{G}_{\overline{\mathcal{X}}}$

Proof in (Pearl, 1995)

# Intuition for Rule 3

$$\Pr(y|\cancel{do(x)}, do(z), w) = \Pr(y|\cancel{do(x)}, w) \quad \text{if} \quad (\mathcal{Y} \perp\!\!\!\perp \mathcal{Z}|\cancel{\mathcal{X}}, \mathcal{W})_{\mathcal{G}_{\cancel{\overline{\mathcal{X}}}\overline{\mathcal{Z}(\mathcal{W})}}}$$

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Suppose

$\Pr(y \mid do(z), w_1, w_2)$
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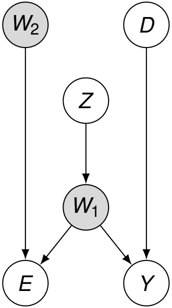


$\mathcal{G}_{\overline{Z}}$

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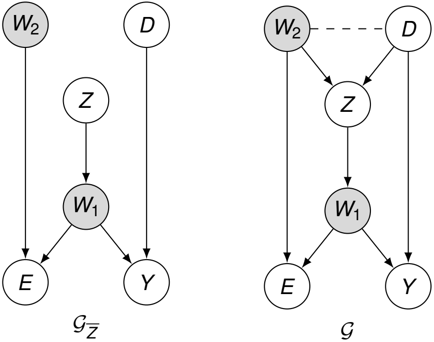

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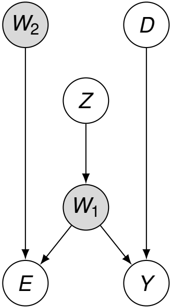
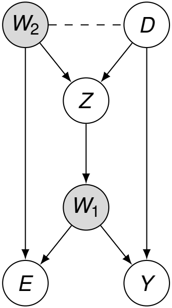



$\mathcal{G}_{\overline{Z}}$

$Y \perp\!\!\!\perp_{\mathcal{G}_{\overline{Z}}} W_2 \mid W_1$

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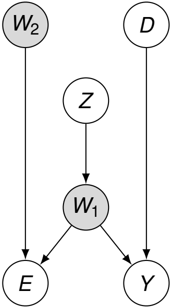


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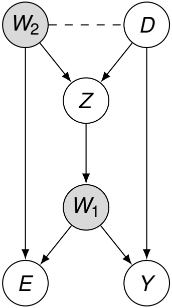


$\mathcal{G}$

$Y \not\perp\!\!\!\perp_{\mathcal{G}} W_2 \mid W_1$

$$\Pr(y|do(z), w) = \Pr(y|w) \quad \text{if} \quad (\mathcal{Y} \perp\!\!\!\perp \mathcal{Z}|\mathcal{W})_{\mathcal{G}_{\overline{\mathcal{Z}(\mathcal{W})}}}$$

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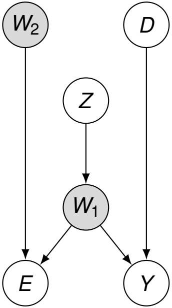


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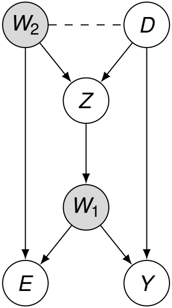



$\mathcal{G}$

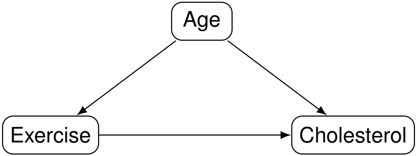
$Y \not\perp\!\!\!\perp_{\mathcal{G}} W_2 \mid W_1$

# Completness of the do-calculus

Theorem A causal effect $P(y \mid do(x))$ is identifiable in a model characterized by a graph $\mathcal{G}$ if and only if there exists a finite sequence of transformations, each conforming to one the Rules 1-3, that reduces $P(y \mid do(x))$ into a standard (i.e., "do"-free) probability expression involving observed quantities.

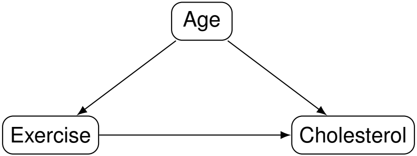
Proof in (Pearl, 1995) and (Shpitser and Pearl, 2006)

# From do-calculus to back-door adjustment




What’s the effect of exercice on cholesterol?

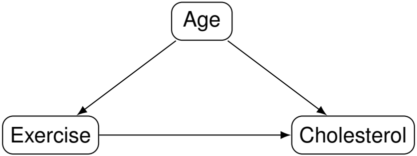
# From do-calculus to back-door adjustment




What's the effect of exercice on cholesterol?

$$\Pr(c \mid do(e)) = \sum_a \Pr(c \mid do(e), a) \Pr(a \mid do(e)) \qquad \text{(Probability Axioms)}$$

# From do-calculus to back-door adjustment

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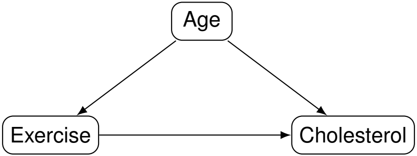
$$\Pr(c \mid do(e)) = \sum_a \Pr(c \mid do(e), a) \Pr(a \mid do(e)) \qquad \text{(Probability Axioms)}$$

$$= \sum_a \Pr(c \mid e, a) \Pr(a \mid do(e)) \qquad \text{(Rule 2)}$$

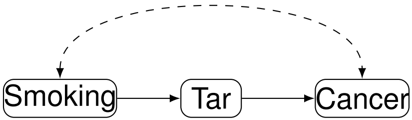
# From do-calculus to back-door adjustment




What's the effect of exercice on cholesterol?

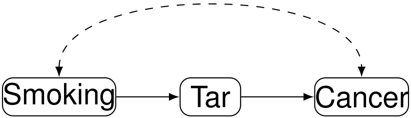
$$\begin{aligned}
\Pr(c \mid do(e)) &= \sum_a \Pr(c \mid do(e), a) \Pr(a \mid do(e)) && \text{(Probability Axioms)} \\
&= \sum_a \Pr(c \mid e, a) \Pr(a \mid do(e)) && \text{(Rule 2)} \\
&= \sum_a \Pr(c \mid e, a) \Pr(a) && \text{(Rule 3)}
\end{aligned}$$

# From do-calculus to front-door adjustment




What's the effect of smoking on cancer?

# From do-calculus to front-door adjustment




What's the effect of smoking on cancer?

$$\Pr(c \mid do(s)) = \sum_t \Pr(c \mid do(s), t) \Pr(t \mid do(s)) \qquad \text{(Probability Axioms)}$$

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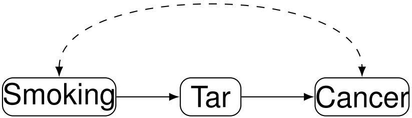


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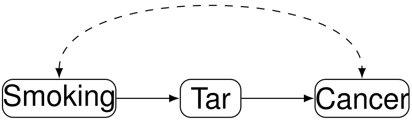


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$$
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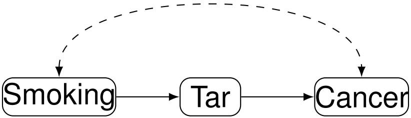
# From do-calculus to front-door adjustment




What's the effect of smoking on cancer?

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\Pr(c \mid do(s)) &= \sum_t \Pr(c \mid do(s), t) \Pr(t \mid do(s)) && \text{(Probability Axioms)} \\
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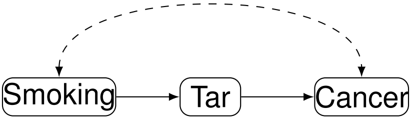


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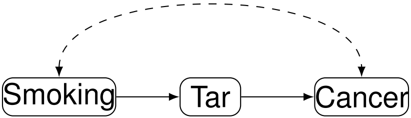


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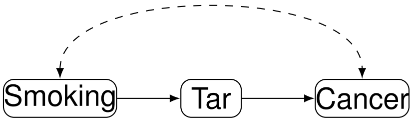
# From do-calculus to front-door adjustment

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# From a calculus toward an automated algorithm

Limitations of the do-calculus:

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- Non-identifiability is complicated

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# From a calculus toward an automated algorithm

Limitations of the do-calculus:

- The do-calculus demands a lot of manual labor
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Is it possible automatize it? Yes! There exists many algorithms.
In this course we will focus on the ID algorithm.

# Table of content

# Some lemmas

Lemma (adding do on non-ancestors)
If

$$\mathcal{W} = (\mathcal{V} \backslash \mathcal{X}) \backslash An(\mathcal{Y})_{\mathcal{G}_{\overline{X}}},$$

then

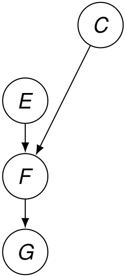
$$\Pr(y \mid do(x)) = \Pr(y \mid do(x), do(w)),$$

where $w$ are arbatrary values of $\mathcal{W}$.

(proof on board)

# Trees and forests

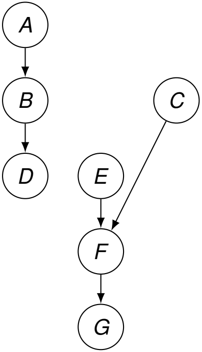
**Tree** A graph $\mathcal{G}$ such that each vertex has at most one child, and only one vertex (called the root) has no children.



# Trees and forests

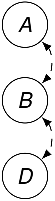
Tree A graph $\mathcal{G}$ such that each vertex has at most one child, and only one vertex (called the root) has no children.

Forest A graph $\mathcal{G}$ such that each vertex has at most one child.



# C-components

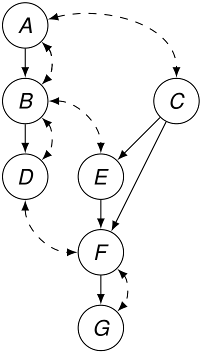
Confounded path A path where all directed arrowheads point at observable vertices, and never away from observable vertices.



# C-components

Confounded path A path where all directed arrowheads point at observable vertices, and never away from observable vertices.

C-component A graph $\mathcal{G}$ where any pair of observable vertices is connected by a confounded path.



# Decomposition into C-components

Any graph can be uniquely partitioned into a collection of subgraphs $C(\mathcal{G})$, each which is a maximal $C$-component.

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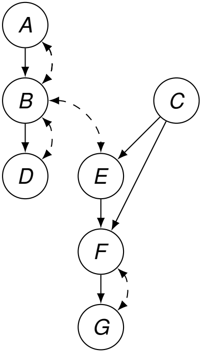
$C(\mathcal{G}) = ?$



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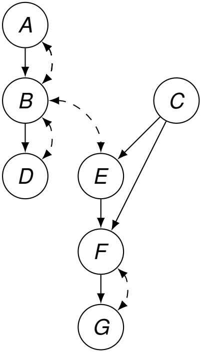
Any graph can be uniquely partitioned into a collection of subgraphs $C(\mathcal{G})$, each which is a maximal $C$-component.

$$C(\mathcal{G}) = \begin{cases} \mathcal{G}[A, B, D, E] \\ \mathcal{G}[C] \\ \mathcal{G}[F, G] \end{cases}$$



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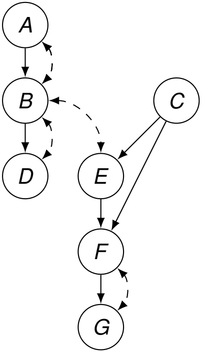



$$C(\mathcal{G}) = \begin{cases} \mathcal{G}[A, B, D, E] \\ \mathcal{G}[C] \\ \mathcal{G}[F, G] \end{cases}$$

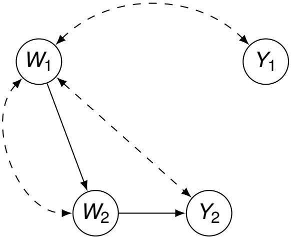
Lemma (c-component factorization) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a causal graph. Let $C(\mathcal{G} \backslash \mathcal{X}) = \{\mathcal{S}_1, \cdots, \mathcal{S}_k\}$. Then

$$\Pr(y \mid do(x)) = \sum_{\mathcal{V} \backslash (y \cup x)} \prod_i \Pr(s_i \mid v \backslash s_i)$$

Proof in (Tian, 2002)

# Hedges

C-forest A graph $\mathcal{G}$ which is both a C-component and a forest. If a given C-forest has a set of root nodes $\mathcal{R}$, we call it $\mathcal{R}$-rooted.



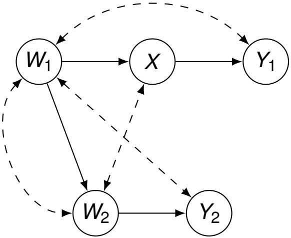
# Hedges

C-forest A graph $\mathcal{G}$ which is both a C-component and a forest. If a given C-forest has a set of root nodes $\mathcal{R}$, we call it $\mathcal{R}$-rooted.

Hedge Let $\mathcal{X}, \mathcal{X} \in \mathcal{V}$ in $\mathcal{G}$. Let $\mathcal{H}, \mathcal{H}'$ be two $R$-rooted C-forests in $\mathcal{G}$ such that

- $\mathcal{H}' \subset \mathcal{H}$,
- $\mathcal{H} \cap X \neq \emptyset$,
- $\mathcal{H}' \cap X = \emptyset$, and
- $R \in An(Y)_{\mathcal{G}_{\underline{X}}}$.

Then $\mathcal{H}$ and $\mathcal{H}'$ form a <u>hedge</u> for $P(y|do(x))$.



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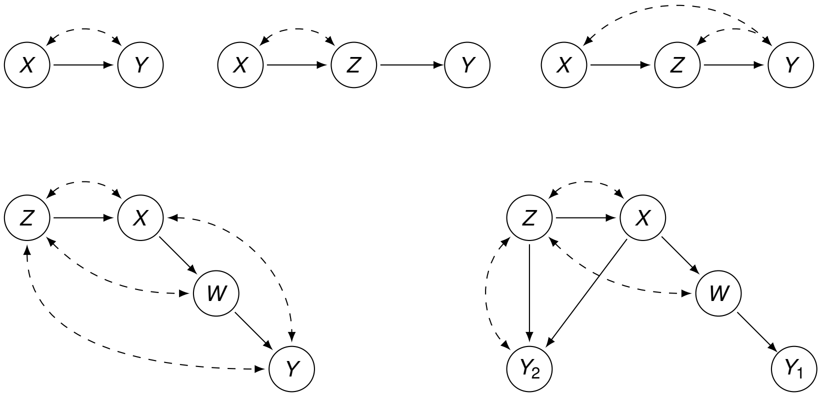

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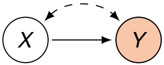


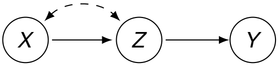


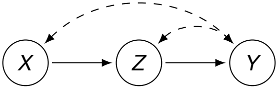


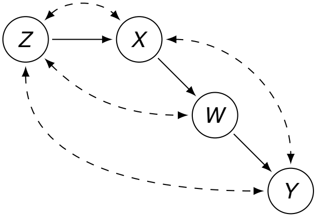


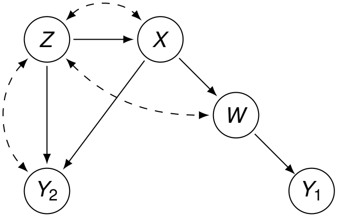

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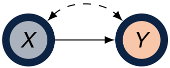


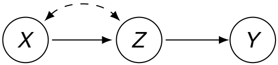


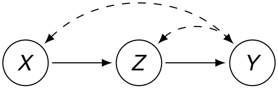


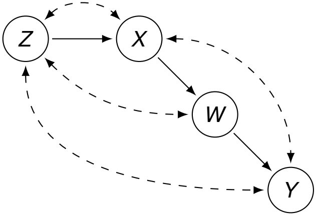


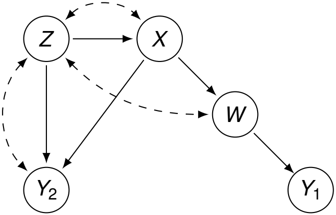

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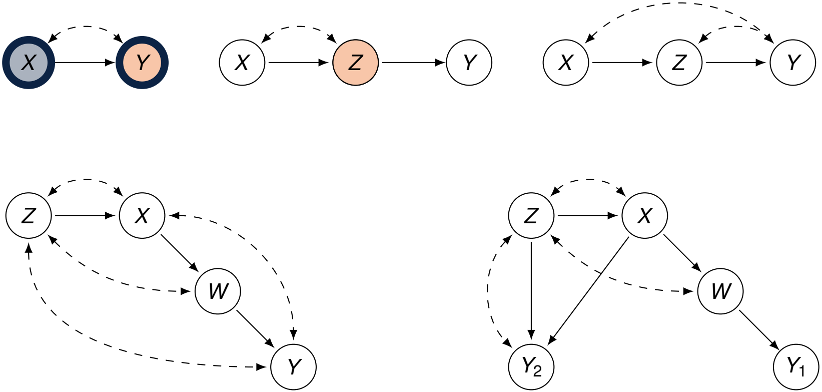

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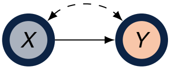


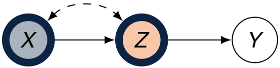


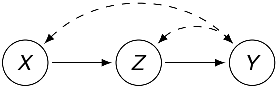


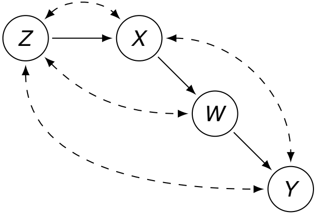


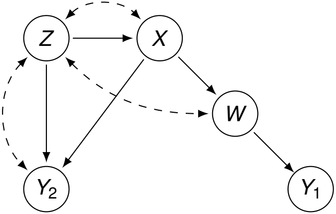

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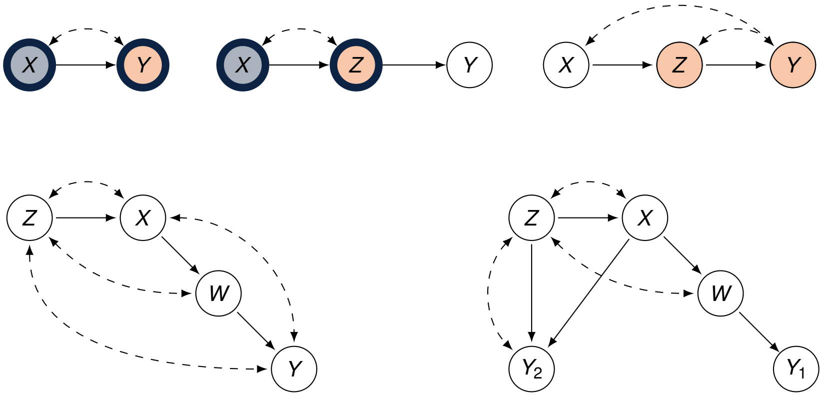

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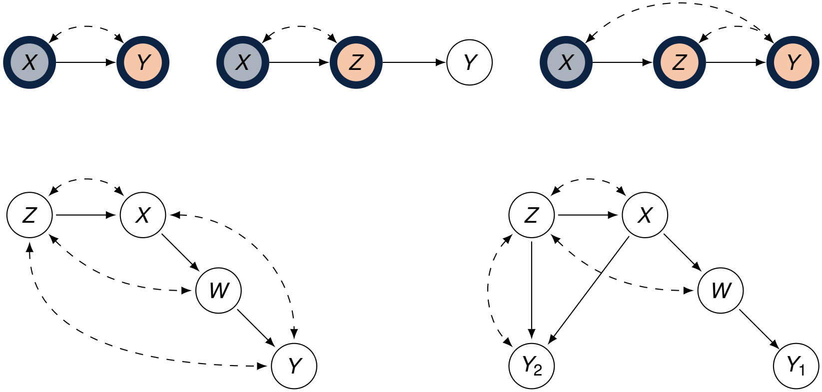

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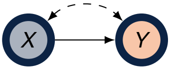


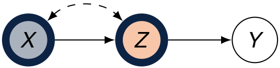


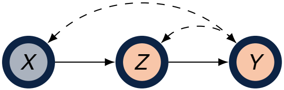


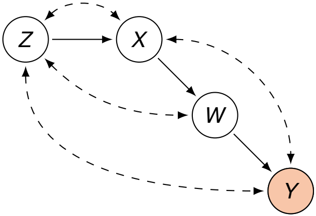


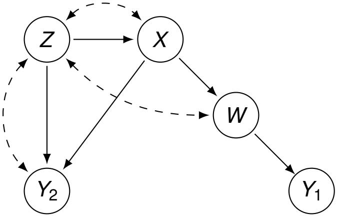

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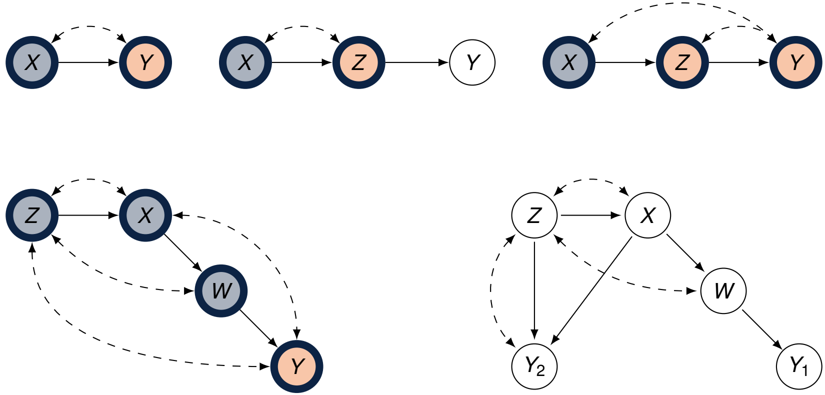

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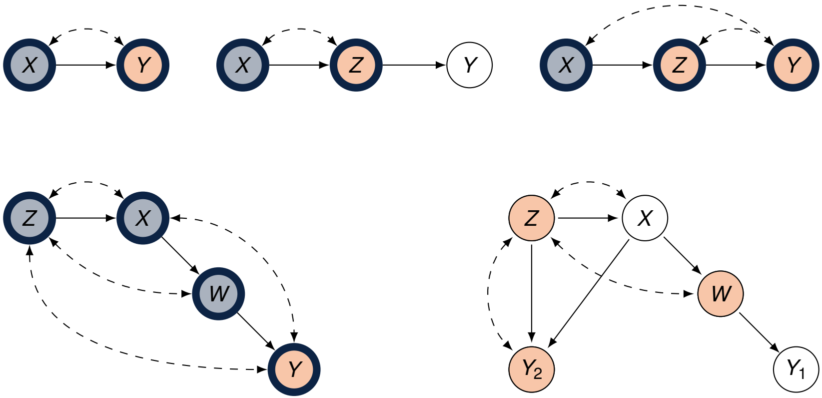

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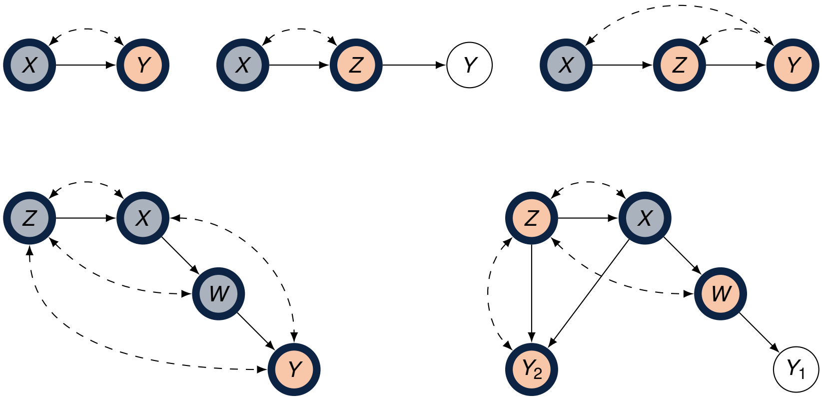

# Hedges and non-identifiability

Theorem (Hedge criterion for non-identifiability) $\Pr(y \mid do(x))$ is not identifiable if and only if $\mathcal{G}$ contains a hedge for some $\Pr(y', do(x'))$, where $\mathcal{Y}' \in \mathcal{Y}$, $\mathcal{X}' \in \mathcal{X}$.

# ID algorithm

**Algorithm 1** ID

**Input:** $\mathcal{Y}, \mathcal{X}, \Pr(\mathcal{V}), \mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or FAIL$(\mathcal{H}, \mathcal{H}')$

**if** $\mathcal{X} = \varnothing$ **then**
  **Return** $\sum_{\mathcal{V} \setminus \mathcal{Y}} \Pr(v)$
**if** $\mathcal{V} \neq An(\mathcal{Y})_{\mathcal{G}}$ **then**
  **Return ID**$(y, x \cap An(\mathcal{Y})_{\mathcal{G}}, \sum_{\mathcal{V} \setminus An(\mathcal{Y})_{\mathcal{G}}} \Pr(v), \mathcal{G}[An(\mathcal{Y})_{\mathcal{G}}])$
**if** $\exists \mathcal{W} = (\mathcal{V} \setminus \mathcal{X}) \setminus An(Y)_{\mathcal{G}_{\overline{X}}}$ such that $\mathcal{W} \neq \varnothing$ **then**
  **Return ID**$(y, x \cup w, P, \mathcal{G})$
**if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}_1, \cdots, \mathcal{S}_k\}$ (for $k \geq 2$) **then**
  **Return** $\sum_{\mathcal{V} \setminus (y \cup x)} \prod_i$ **ID**$(s_i, v \setminus s_i, \Pr(v), \mathcal{G})$
**else if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}\}$ **then**
  **if** $C(\mathcal{G}) = \{\mathcal{G}\}$ **then**
    **Return FAIL**$(\mathcal{G}, \mathcal{S})$
  **if** $S \in C(\mathcal{G})$ **then**
    **Return** $\sum_{s \setminus y} \prod_{V_i \in S} \Pr(v_i | v_{\pi}^{(i-1)})$
  **if** $\exists S', S \subseteq S' \in C(\mathcal{G})$ **then**
    **Return ID**$(y, x \cap S', \prod_{V_i \in S'} \Pr(V_i \mid V_{\pi}^{(i-1)} \cap S', v_{\pi}^{(i-1)} \setminus S'), S')$

# ID algorithm

**Algorithm 2** ID

**Input:** $\mathcal{Y}, \mathcal{X}, \Pr(\mathcal{V}), \mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or $\text{FAIL}(\mathcal{H}, \mathcal{H}')$

**if** $\mathcal{X} = \varnothing$ **then**
  **Return** $\sum_{\mathcal{V} \setminus \mathcal{Y}} \Pr(v)$
**if** $\mathcal{V} \neq An(\mathcal{Y})_{\mathcal{G}}$ **then**
  **Return ID**$(y, x \cap An(\mathcal{Y})_{\mathcal{G}}, \sum_{\mathcal{V} \setminus An(\mathcal{Y})_{\mathcal{G}}} \Pr(v), \mathcal{G}[An(\mathcal{Y})_{\mathcal{G}}])$
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**else if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}\}$ **then**
  **if** $C(\mathcal{G}) = \{\mathcal{G}\}$ **then**
    **Return FAIL**$(\mathcal{G}, \mathcal{S})$
  **if** $S \in C(\mathcal{G})$ **then**
    **Return** $\sum_{s \setminus y} \prod_{V_i \in S} \Pr(v_i | v_{\pi}^{(i-1)})$
  **if** $\exists S', S \subseteq S' \in C(\mathcal{G})$ **then**
    **Return ID**$(y, x \cap S', \prod_{V_i \in S'} \Pr(V_i \mid V_{\pi}^{(i-1)} \cap S', v_{\pi}^{(i-1)} \setminus S'), S')$

Trivial

**Algorithm 3** ID

**Input:** $\mathcal{Y}$, $\mathcal{X}$, $\Pr(\mathcal{V})$, $\mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or FAIL$(\mathcal{H}, \mathcal{H}')$

**if** $\mathcal{X} = \varnothing$ **then**
    **Return** $\sum_{\mathcal{V}\setminus\mathcal{Y}} \Pr(v)$
**if** $\mathcal{V} \neq An(\mathcal{Y})_{\mathcal{G}}$ **then**
    **Return ID**$(y, x \cap An(\mathcal{Y})_{\mathcal{G}}, \sum_{\mathcal{V}\setminus An(\mathcal{Y})_{\mathcal{G}}} \Pr(v), \mathcal{G}[An(\mathcal{Y})_{\mathcal{G}}])$
**if** $\exists \mathcal{W} = (\mathcal{V}\setminus\mathcal{X})\setminus An(Y)_{\mathcal{G}_{\overline{X}}}$ such that $\mathcal{W} \neq \varnothing$ **then**
    **Return ID**$(y, x \cup w, P, \mathcal{G})$
**if** $C(\mathcal{G}\setminus\mathcal{X}) = \{\mathcal{S}_1, \cdots, \mathcal{S}_k\}$ (for $k \geq 2$) **then**
    **Return** $\sum_{\mathcal{V}\setminus(y\cup x)} \prod_i$ **ID**$(s_i, v\setminus s_i, \Pr(v), \mathcal{G})$
**else if** $C(\mathcal{G}\setminus\mathcal{X}) = \{\mathcal{S}\}$ **then**
    **if** $C(\mathcal{G}) = \{\mathcal{G}\}$ **then**
        **Return FAIL**$(\mathcal{G}, \mathcal{S})$
    **if** $S \in C(\mathcal{G})$ **then**
        **Return** $\sum_{s\setminus y} \prod_{V_i \in S} \Pr(v_i | v_\pi^{(i-1)})$
    **if** $\exists S', S \subseteq S' \in C(\mathcal{G})$ **then**
        **Return ID**$(y, x \cap S', \prod_{V_i \in S'} \Pr(V_i \mid V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)}\setminus S'), S')$

Trivial

# ID algorithm

**Algorithm 4** ID

**Input:** $\mathcal{Y}, \mathcal{X}, \Pr(\mathcal{V}), \mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or $\text{FAIL}(\mathcal{H}, \mathcal{H}')$

**if** $\mathcal{X} = \varnothing$ **then**
  **Return** $\sum_{\mathcal{V} \setminus \mathcal{Y}} \Pr(v)$
**if** $\mathcal{V} \neq An(\mathcal{Y})_{\mathcal{G}}$ **then**
  **Return ID**$(y, x \cap An(\mathcal{Y})_{\mathcal{G}}, \sum_{\mathcal{V} \setminus An(\mathcal{Y})_{\mathcal{G}}} \Pr(v), \mathcal{G}[An(\mathcal{Y})_{\mathcal{G}}])$
**if** $\exists \mathcal{W} = (\mathcal{V} \setminus \mathcal{X}) \setminus An(Y)_{\mathcal{G}_{\overline{X}}}$ such that $\mathcal{W} \neq \varnothing$ **then**
  **Return ID**$(y, x \cup w, P, \mathcal{G})$
**if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}_1, \cdots, \mathcal{S}_k\}$ (for $k \geq 2$) **then**
  **Return** $\sum_{\mathcal{V} \setminus (y \cup x)} \prod_i$ **ID**$(s_i, v \setminus s_i, \Pr(v), \mathcal{G})$
**else if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}\}$ **then**
  **if** $C(\mathcal{G}) = \{\mathcal{G}\}$ **then**
    **Return FAIL**$(\mathcal{G}, \mathcal{S})$
  **if** $S \in C(\mathcal{G})$ **then**
    **Return** $\sum_{s \setminus y} \prod_{V_i \in S} \Pr(v_i | v_\pi^{(i-1)})$
  **if** $\exists S', S \subseteq S' \in C(\mathcal{G})$ **then**
    **Return ID**$(y, x \cap S', \prod_{V_i \in S'} \Pr(V_i \mid V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S'), S')$

Lemma (adding do on non-ancestors)

# ID algorithm

**Algorithm 5** ID

**Input:** $\mathcal{Y}$, $\mathcal{X}$, $\Pr(\mathcal{V})$, $\mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or FAIL$(\mathcal{H}, \mathcal{H}')$

**if** $\mathcal{X} = \varnothing$ **then**

  **Return** $\sum_{\mathcal{V}\setminus\mathcal{Y}} \Pr(v)$

**if** $\mathcal{V} \neq An(\mathcal{Y})_{\mathcal{G}}$ **then**

  **Return ID**$(y, x \cap An(\mathcal{Y})_{\mathcal{G}}, \sum_{\mathcal{V}\setminus An(\mathcal{Y})_{\mathcal{G}}} \Pr(v), \mathcal{G}[An(\mathcal{Y})_{\mathcal{G}}])$

**if** $\exists \mathcal{W} = (\mathcal{V}\setminus\mathcal{X})\setminus An(Y)_{\mathcal{G}_{\overline{X}}}$ such that $\mathcal{W} \neq \varnothing$ **then**

  **Return ID**$(y, x \cup w, P, \mathcal{G})$

**if** $C(\mathcal{G}\setminus\mathcal{X}) = \{\mathcal{S}_1, \cdots, \mathcal{S}_k\}$ (for $k \geq 2$) **then**

  **Return** $\sum_{\mathcal{V}\setminus(y\cup x)} \prod_i$ **ID**$(s_i, v\setminus s_i, \Pr(v), \mathcal{G})$

**else if** $C(\mathcal{G}\setminus\mathcal{X}) = \{\mathcal{S}\}$ **then**

  **if** $C(\mathcal{G}) = \{\mathcal{G}\}$ **then**

    **Return FAIL**$(\mathcal{G}, \mathcal{S})$

  **if** $S \in C(\mathcal{G})$ **then**

    **Return** $\sum_{s\setminus y} \prod_{V_i \in S} \Pr(v_i | v_\pi^{(i-1)})$

  **if** $\exists S', S \subseteq S' \in C(\mathcal{G})$ **then**

    **Return ID**$(y, x \cap S', \prod_{V_i \in S'} \Pr(V_i \mid V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S'), S')$

Lemma (c-component factorization)

# ID algorithm

**Algorithm 6** ID

**Input:** $\mathcal{Y}$, $\mathcal{X}$, $\Pr(\mathcal{V})$, $\mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or FAIL$(\mathcal{H}, \mathcal{H}')$

**if** $\mathcal{X} = \varnothing$ **then**
    **Return** $\sum_{\mathcal{V} \setminus \mathcal{Y}} \Pr(v)$
**if** $\mathcal{V} \neq An(\mathcal{Y})_{\mathcal{G}}$ **then**
    **Return ID**$(y, x \cap An(\mathcal{Y})_{\mathcal{G}}, \sum_{\mathcal{V} \setminus An(\mathcal{Y})_{\mathcal{G}}} \Pr(v), \mathcal{G}[An(\mathcal{Y})_{\mathcal{G}}])$
**if** $\exists \mathcal{W} = (\mathcal{V} \setminus \mathcal{X}) \setminus An(Y)_{\mathcal{G}_{\overline{X}}}$ such that $\mathcal{W} \neq \varnothing$ **then**
    **Return ID**$(y, x \cup w, P, \mathcal{G})$
**if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}_1, \cdots, \mathcal{S}_k\}$ (for $k \geq 2$) **then**
    **Return** $\sum_{\mathcal{V} \setminus (y \cup x)} \prod_i$ **ID**$(s_i, v \setminus s_i, \Pr(v), \mathcal{G})$
**else if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}\}$ **then**
    **if** $C(\mathcal{G}) = \{\mathcal{G}\}$ **then**
        **Return FAIL**$(\mathcal{G}, \mathcal{S})$
    **if** $S \in C(\mathcal{G})$ **then**
        **Return** $\sum_{s \setminus y} \prod_{V_i \in S} \Pr(v_i | v_\pi^{(i-1)})$
    **if** $\exists S', S \subseteq S' \in C(\mathcal{G})$ **then**
        **Return ID**$(y, x \cap S', \prod_{V_i \in S'} \Pr(V_i \mid V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S'), S')$

# ID algorithm

**Algorithm 7** ID

**Input:** $\mathcal{Y}$, $\mathcal{X}$, $\Pr(\mathcal{V})$, $\mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or FAIL$(\mathcal{H}, \mathcal{H}')$

**if** $\mathcal{X} = \varnothing$ **then**
  **Return** $\sum_{\mathcal{V} \setminus \mathcal{Y}} \Pr(v)$
**if** $\mathcal{V} \neq An(\mathcal{Y})_{\mathcal{G}}$ **then**
  **Return ID**$(y, x \cap An(\mathcal{Y})_{\mathcal{G}}, \sum_{\mathcal{V} \setminus An(\mathcal{Y})_{\mathcal{G}}} \Pr(v), \mathcal{G}[An(\mathcal{Y})_{\mathcal{G}}])$
**if** $\exists \mathcal{W} = (\mathcal{V} \setminus \mathcal{X}) \setminus An(Y)_{\mathcal{G}_{\overline{X}}}$ such that $\mathcal{W} \neq \varnothing$ **then**
  **Return ID**$(y, x \cup w, P, \mathcal{G})$
**if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}_1, \cdots, \mathcal{S}_k\}$ (for $k \geq 2$) **then**
  **Return** $\sum_{\mathcal{V} \setminus (y \cup x)} \prod_i$ **ID**$(s_i, v \setminus s_i, \Pr(v), \mathcal{G})$
**else if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}\}$ **then**
  **if** $C(\mathcal{G}) = \{\mathcal{G}\}$ **then**
    **Return FAIL**$(\mathcal{G}, \mathcal{S})$
  **if** $S \in C(\mathcal{G})$ **then**
    **Return** $\sum_{s \setminus y} \prod_{V_i \in S} \Pr(v_i | v_\pi^{(i-1)})$
  **if** $\exists S', S \subseteq S' \in C(\mathcal{G})$ **then**
    **Return ID**$(y, x \cap S', \prod_{V_i \in S'} \Pr(V_i \mid V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S'), S')$

Theorem (Hedge criterion for non-identifiability)

# ID algorithm

**Algorithm 8** ID

**Input:** $\mathcal{Y}$, $\mathcal{X}$, $\Pr(\mathcal{V})$, $\mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or FAIL$(\mathcal{H}, \mathcal{H}')$

```
if 𝒳 = ∅ then
    Return ∑_{𝒱\𝒴} Pr(v)
if 𝒱 ≠ An(𝒴)_𝒢 then
    Return ID(y, x ∩ An(𝒴)_𝒢, ∑_{𝒱\An(𝒴)_𝒢} Pr(v), 𝒢[An(𝒴)_𝒢])
if ∃𝒲 = (𝒱\𝒳)\An(Y)_{𝒢_X̄} such that 𝒲 ≠ ∅ then
    Return ID(y, x ∪ w, P, 𝒢)
if C(𝒢\𝒳) = {𝒮_1, ⋯, 𝒮_k} (for k ≥ 2) then
    Return ∑_{𝒱\(y∪x)} ∏_i ID(s_i, v\s_i, Pr(v), 𝒢)
else if C(𝒢\𝒳) = {𝒮} then
    if C(𝒢) = {𝒢} then
        Return FAIL(𝒢, 𝒮)
    if S ∈ C(𝒢) then
        Return ∑_{s∖y} ∏_{V_i∈S} Pr(v_i | v_π^(i−1))
    if ∃S′, S ⊆ S′ ∈ C(𝒢) then
        Return ID(y, x ∩ S′, ∏_{V_i∈S′} Pr(V_i | V_π^(i−1) ∩ S′, v_π^(i−1)\S′), S′)
```

Proof in (Shpitser and Pearl, 2006)

# ID algorithm

**Algorithm 9** ID

**Input:** $\mathcal{Y}$, $\mathcal{X}$, $\Pr(\mathcal{V})$, $\mathcal{G}$

**Output:** *do*-free expression for $\Pr(y \mid do(x))$ or FAIL$(\mathcal{H}, \mathcal{H}')$

**if** $\mathcal{X} = \varnothing$ **then**
    **Return** $\sum_{\mathcal{V} \setminus \mathcal{Y}} \Pr(v)$
**if** $\mathcal{V} \neq An(\mathcal{Y})_{\mathcal{G}}$ **then**
    **Return ID**$(y, x \cap An(\mathcal{Y})_{\mathcal{G}}, \sum_{\mathcal{V} \setminus An(\mathcal{Y})_{\mathcal{G}}} \Pr(v), \mathcal{G}[An(\mathcal{Y})_{\mathcal{G}}])$
**if** $\exists \mathcal{W} = (\mathcal{V} \setminus \mathcal{X}) \setminus An(Y)_{\mathcal{G}_{\overline{X}}}$ such that $\mathcal{W} \neq \varnothing$ **then**
    **Return ID**$(y, x \cup w, P, \mathcal{G})$
**if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}_1, \cdots, \mathcal{S}_k\}$ (for $k \geq 2$) **then**
    **Return** $\sum_{\mathcal{V} \setminus (y \cup x)} \prod_i$ **ID**$(s_i, v \setminus s_i, \Pr(v), \mathcal{G})$
**else if** $C(\mathcal{G} \setminus \mathcal{X}) = \{\mathcal{S}\}$ **then**
    **if** $C(\mathcal{G}) = \{\mathcal{G}\}$ **then**
        **Return FAIL**$(\mathcal{G}, \mathcal{S})$
    **if** $S \in C(\mathcal{G})$ **then**
        **Return** $\sum_{s \setminus y} \prod_{V_i \in S} \Pr(v_i | v_\pi^{(i-1)})$
    **if** $\exists S', S \subseteq S' \in C(\mathcal{G})$ **then**
        **Return ID**$(y, x \cap S', \prod_{V_i \in S'} \Pr(V_i \mid V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S'), S')$

Proof in (Shpitser and Pearl, 2006)

# Completeness of ID algorithm

Theorem (Soudness of the ID algorithm) Whenever the ID algorithm returns an expression for $\Pr(y \mid do(x))$, it is correct.

Partially proved in the previous slides.

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Theorem (Soudness of the ID algorithm) Whenever the ID algorithm returns an expression for $\Pr(y \mid do(x))$, it is correct.

Partially proved in the previous slides.

Theorem (Completeness of ID algorithm) ID is complete.

Proof in (Shpitser and Pearl, 2006)

# Table of content

# Conclusion

- do calculus is complete;
- The ID algorithm is complete.

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# Some extensions

- The IDC algorithm that support conditioning;
- Finding optimal adjustment sets;
- Identifiability for direct effects and indirect effects.

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- The IDC algorithm that support conditioning;
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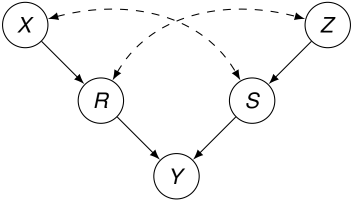
# References

## Direct inspirations


1. *Causal diagrams for empirical research*, J. Pearl. Biometrika, 1995
2. *Identification of Joint Interventional Distributions in Recursive Semi-Markovian Causal Models*, I. Shpitser, J. Pearl. Proceedings of the Twenty National Conference on Artificial Intelligence, 2006
3. *Complete Identification Methods for the Causal Hierarchy*, I. Shpitser, J. Pearl. Journal of Machine Learning Research, 2008
4. *Studies in Causal Reasoning and Learning*, J. Tian. PhD thesis, 2002
5. *Causality*, J. Pearl. Cambridge University Press, 2nd edition, 2009

# Exercise 1.1

Consider the following semi-Markovian model:



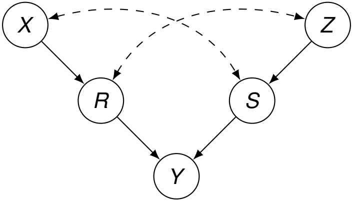

Test, using do-calculus, whether the causal effect

$$P(y \mid do(r))$$

is identifiable. If the answer is yes, provide an expression for it that does not contain the *do*() operator.

# Exercise 1.2

Consider the following semi-Markovian model:



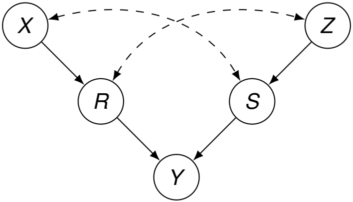

Test, using do-calculus, whether the causal effect

$$P(r \mid do(y))$$

is identifiable. If the answer is yes, provide an expression for it that does not contain the *do*() operator.

# Exercise 1.3

Consider the following semi-Markovian model:



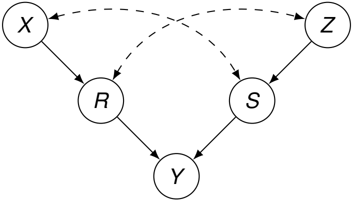

Test, using do-calculus, whether the causal effect

$$P(y \mid do(r), do(s))$$

is identifiable. If the answer is yes, provide an expression for it that does not contain the *do*() operator.

# Exercise 1.4

Consider the following semi-Markovian model:



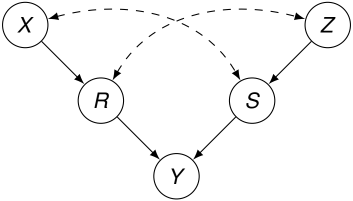

Test, using do-calculus, whether the causal effect

$$P(r \mid do(x), do(z))$$

is identifiable. If the answer is yes, provide an expression for it that does not contain the *do*() operator.

# Exercise 1.5

Consider the following semi-Markovian model:



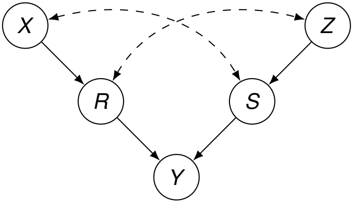

Test, using do-calculus, whether the causal effect

$$P(s \mid do(x), do(z))$$

is identifiable. If the answer is yes, provide an expression for it that does not contain the *do*() operator.

# Exercise 1.6

Consider the following semi-Markovian model:



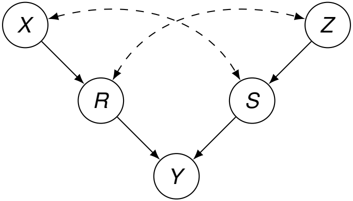

Test, using do-calculus, whether the causal effect

$$P(r, s \mid do(x), do(z))$$

is identifiable. If the answer is yes, provide an expression for it that does not contain the *do*() operator.

# Exercise 1.7

Consider the following semi-Markovian model:



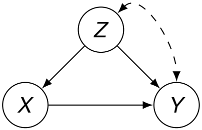

Test, using do-calculus, whether the causal effect

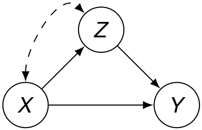
$$P(y \mid do(x), do(z))$$

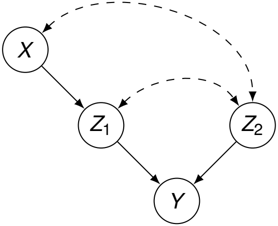
is identifiable. If the answer is yes, provide an expression for it that does not contain the *do*( ) operator.

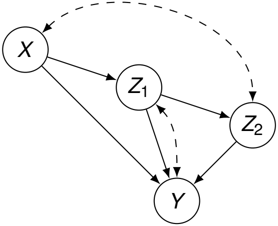
# Exercise 2

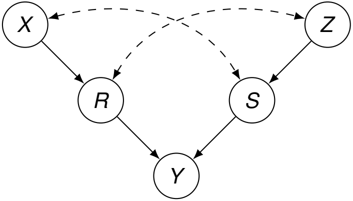
Which of the following semi-Markovian models admet an identifiable causal effect $\Pr(y \mid do(x))$?



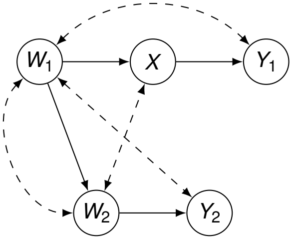



# Exercise 3

Consider the following semi-Markovian model containing a hedge for $\Pr(y \mid do(x))$:




- Is it possible to remove the hedge by adding one directed edge to the graph? If yes, which one?
- Is it possible to remove the hedge by deleting one directed edge from the graph? If yes, which one?