Introduction to causal graphical models

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Causality

Does Obesity Shorten Life? Or is it the Soda? (Pearl, 2018)

Many applications in machine learning, medecine (science in general), root cause analysis, ...

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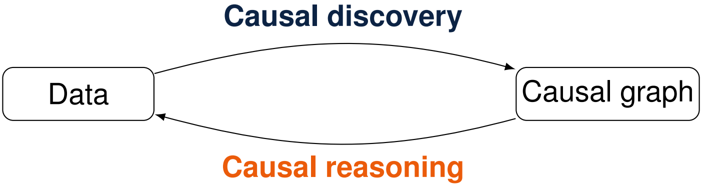

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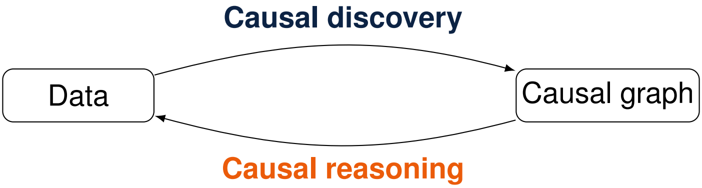

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Causes and effects

The same causes produce the same effects ..., do they?

- Smoking causes lung cancer
- The sound of your alarm makes you wake up
- Cause: I flipped the light switch - Effect: the light came on

Probabilities are used to capture uncertainty/indeterminacy

→ Probabilistic Causal Models

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(conditional) Independence

Conditional independence of random variables For a distribution P, X and Y are independent conditioned on Z, noted $X \perp\!\!\!\perp_P Y \mid Z$, if:

$$P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z) \; (or \; P(X \mid Y, Z) = P(X \mid Z) \; if \; P(Y, Z) > 0)$$

Illustration

- $Z \sim Bi(9, 0.5)$, $X \mid Z = z \sim \mathcal{N}(z, 1)$ and $Y \mid Z = z \sim \mathcal{N}(z, 1)$
- $Z \sim Bi(3, 0.5)$, $X \sim Exp(1)$ and $Y \mid X = x \sim 0.15\delta_0 + 0.85 Pois(x)$

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Probabilistic causal models (1)

A tuple $M = \langle(\mathcal{U}, \mathcal{V}, \mathcal{F}, P(\mathcal{U}))\rangle$ with

1. $\mathcal{U}$ is a set of unobserved background variables which can't be manipulated
2. $\mathcal{V} = \{X_1, ..., X_n\}$ is a set of observed variables
3. $\mathcal{F}$ is a set of functions s.t. f_i $(1 \leq i \leq n)$ specifies X_i: $X_i = f_i(\mathcal{E}_i)$ with $\mathcal{E}_i \subseteq \mathcal{U} \cup \mathcal{V}$
4. $P(\mathcal{U})$ is a joint distribution over $\mathcal{U}$

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Probabilistic causal models (2)

$P(\mathcal{U})$ and $\mathcal{F}$ induce a joint distribution over $\mathcal{V}$:

$$\begin{aligned} P(\mathcal{V}) &= \sum_{u \in D_U} P(\mathcal{V}, u) \\ &= \sum_{u \in D_U} P(\mathcal{V} \mid u) P(u) \\ &= \sum_{u \in D_U} \prod_{i=1}^{n} P(X_i \mid X_1, ..., X_{i-1}, u) P(u) \end{aligned}$$

- More interesting factorizations?
- What about $P(u)$?

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Probabilistic causal models (3)

Induced graph *The graph $\mathcal{G}(M)$ induced by a probabilistic causal model M has vertices $\mathcal{V}$ and an edge $X_i \rightarrow X_j$ whenever f_i depends on X_j. In addition, G contains a bidirected edge, denoted $X_i \leftarrow\!\!-\!\!-\!\!\rightarrow X_j$, whenever f_i and f_j depend on a common subset of $\mathcal{U}$*

Markovian causal model *A causal model M is Markovian if the graph induced by M contains no bidirected edges (causal sufficiency)*

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&= \sum_{u \in D_U} \prod_{i=1}^{n} \frac{P(X_i, u_i \mid X_1, \ldots, X_{i-1})}{P(u_i)} P(u_i) \\
&= \sum_{u_1, u_2, \ldots, u_n} \prod_{i=1}^{n} P(X_i, u_i \mid X_1, \ldots, X_{i-1}) \\
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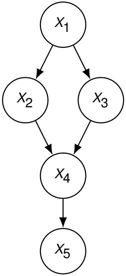
Example

X_1: season (can take on 4 values)
X_2: rain (binary yes/no)
X_3: sprinkler (binary on/off)
X_4: wet (binary yes/no)
X_5: slippery (binary yes/no)

Which causal graph should we consider?

Example (cont'd)

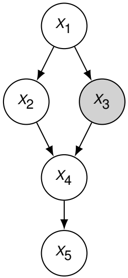


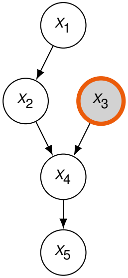
With no confounders:

$$P(\mathcal{V}) = P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_1) \\ P(X_4 \mid X_2, X_3)P(X_5 \mid X_4)$$

Example (cont'd)

Conditioning vs intervention



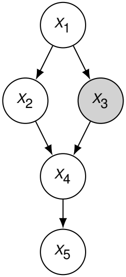


Example (cont'd)

Conditioning

$$P(X_1, X_2, X_4, X_5 \mid X_3 = off) = \frac{P(X_1, X_2, X_4, X_5, X_3 = off)}{\sum_{x_1} P(X_1 = x_1)P(X_3 = off \mid X_1 = x_1)}$$

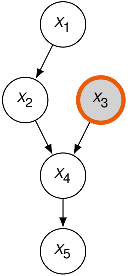
$$= \frac{P(X_1)P(X_2 \mid X_1)P(X_3 = off \mid X_1)P(X_4 \mid X_2, X_3 = off)P(X_5 \mid X_4)}{\sum_{x_1} P(X_1 = x_1)P(X_3 = off \mid X_1 = x_1)}$$



Example (cont'd)

Intervention

$$P_{X_3=off}(X_1, X_2, X_4, X_5) = P(X_1)P(X_2 \mid X_1)P(X_4 \mid X_2, X_3 = off)P(X_5 \mid X_4)$$



Example (cont'd)

Conditioning vs intervention

$P(X_1, X_2, X_4, X_5 \mid X_3 = off)$ vs $P_{X_3=off}(X_1, X_2, X_4, X_5)$

Identification (identifiability)

Example (cont'd)

Conditioning vs intervention

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Identification (identifiability)

Interventional theory

Causation in the interventional theory

- *A* causes *B* if and only if there is a possible intervention on *A* which changes *B*
- An intervention on *A* must completely disrupt the causal relation between *A* and its previous causes so that the value of *A* is entirely fixed by this intervention

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Causal discovery vs causal inference

Causal discovery From observational data, infer causal graph with or without hidden confounders (hidden common causes) - Sessions 3 and 4

Causal inference Reasoning on the causal graph through interventions (and asking counterfactual questions) - Sessions 6 and 7

In the remainder:

- Focus on directed acyclic graphs
- Understand the relationships between graphs and distributions

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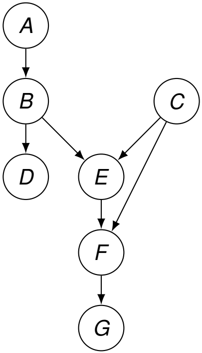
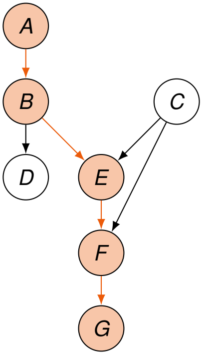
Basic graph concepts

Let us consider the following graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:



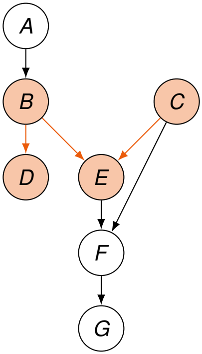
Basic graph concepts (cont'd)

Directed path: $A \to B \to E \to F \to G$ ($A \rightsquigarrow G$)



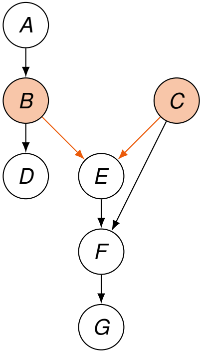
Basic graph concepts (cont'd)

Path (trail): $D \leftarrow B \rightarrow E \leftarrow C$ ($D \circ\!\!\sim\!\!\sim\!\!\sim\!\!\circ C$)



Basic graph concepts (cont'd)

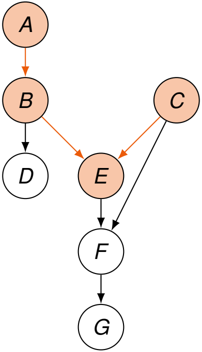
Parents, ancestors: $Pa(E) = \{B, C\}$,



Basic graph concepts (cont'd)

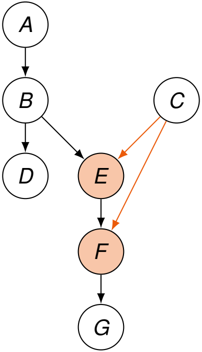
Parents, ancestors: $Pa(E) = \{B, C\}$, $An(E) = \{A, B, C, E\}$

An: transitive closure of the parents relation



Basic graph concepts (cont'd)

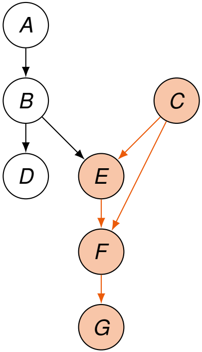
Children, descendants: $Ch(C) = \{E, F\}$,



Basic graph concepts (cont'd)

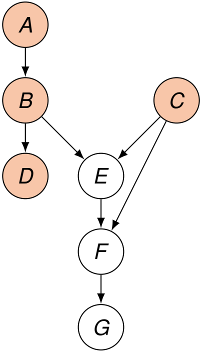
Children, descendants: $Ch(C) = \{E, F\}$, $De(C) = \{C, E, F, G\}$

De: transitive closure of the children relation



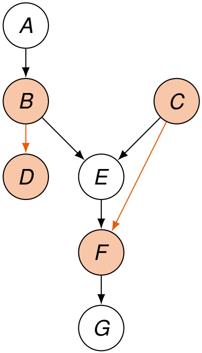
Basic graph concepts (cont'd)

Upwards-closed sets: a subset of nodes $\mathcal{S}$ is upward-closed (or ancestral) if $\forall S \in \mathcal{S}, An(S) \subseteq \mathcal{S}$



Basic graph concepts (cont'd)

Induced subgraph $\mathcal{G}[\mathcal{S}]$: $\mathcal{G}[\{B, C, D, F\}]$



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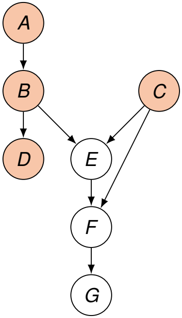
Bayesian networks and compatibility

A Bayesian network is a DAG (directed acyclic graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *along with a joint distribution* $P(\mathcal{V})$ *that admits the factorization* $P(\mathcal{V}) = \prod_{X \in \mathcal{V}} P(X \mid Pa_G(X))$

Compatibility We say that a distribution $P(\mathcal{V})$ is compatible with (or Markov relative to) a DAG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $P(\mathcal{V}) = \prod_{X \in \mathcal{V}} P(X \mid Pa(X))$. We denote by $\mathcal{P}(\mathcal{V})$ the set of distributions compatible with $\mathcal{G}$.

Observation

Upwards-closed set If P is compatible with $\mathcal{G}$ and $\mathcal{S} \subseteq \mathcal{V}$ is upwards-closed, then $P(\mathcal{S})$ is compatible with $\mathcal{G}[\mathcal{S}]$, *i.e.*, $P(\mathcal{S}) = \prod_{S \in \mathcal{S}} P(S \mid Pa(S))$ (proof on board)

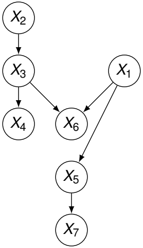


Markov conditions

Ordered Markov condition P is compatible with $\mathcal{G}$ *iff* in any topological ordering each X_i is independent of its predecessors given its parents (proof on board)

Topological ordering: for any edge $X_i \rightarrow X_j$, $i < j$

Parental Markov condition P is compatible with $\mathcal{G}$ *iff* every variable is independent of its non-descendants given its parents (proof on board)



Conditioning on common ancestors

Property For disjoint $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{V}$, if $An(\mathcal{X}) \cap An(\mathcal{Y}) \subseteq \mathcal{Z}$ and $An(\mathcal{Z}) \subseteq (\mathcal{Z})$, then

$$P(\mathcal{X}, \mathcal{Y} \mid \mathcal{Z}) = P(\mathcal{X} \mid \mathcal{Z}) P(\mathcal{Y} \mid \mathcal{Z}) \text{ (i.e., } \mathcal{X} \perp\!\!\!\perp_P \mathcal{Y} \mid \mathcal{Z})$$

in any distribution P compatible with $\mathcal{G}$
(proof on board)

Illustration

Causal Bayesian networks

Causal Markov condition Every Markovian causal model M induces a distribution that is compatible with the induced graph $\mathcal{G}[M]$

Causal Bayesian network (Pearl 2000) Let $P(\mathcal{V})$ be a probability distribution and let $P_s(\mathcal{V})$ denote the distribution resulting from the intervention that sets a subset $\mathcal{S}$ of variables to constants s. Let $\mathcal{P}_*$ denote the set of all interventional distributions $P_s(\mathcal{V})$. A DAG $\mathcal{G}$ is said to be a *causal Bayesian network* compatible with $\mathcal{P}_*$ *iff* for every $P_s(\mathcal{V}) \in \mathcal{P}_*$:

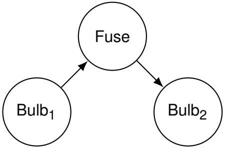
(i) $P_s(\mathcal{V})$ is Markov relative to $\mathcal{G}$

(ii) $P_s(s_i) = 1$ or all $S_i \in \mathcal{S}$ whenever s_i is consistent with $\mathcal{S} = s$

(ii) $P_s(x_i \mid Pa(X_i)) = P(x_i \mid Pa(X_i))$ for all $X_i \notin \mathcal{S}$ whenever $Pa(X_i)$ is consistent with $\mathcal{S} = s$

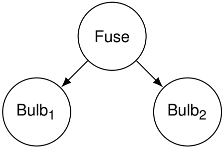
Causal Bayesian networks: example

Bayesian networks vs causal graph



Non causal graph

Bulb$_1$ $\perp\!\!\!\perp_P$ Bulb$_2$ | Fuse



Causal graph

Bulb$_1$ $\perp\!\!\!\perp_P$ Bulb$_2$ | Fuse

- $F \sim \mathcal{U}\{0, 1\}$
- $P(Bulb_1 = 1 \mid Fuse = 1) = 1 - \epsilon_1,\ P(Bulb_1 = 0 \mid Fuse = 1) = \epsilon_1$
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- $\epsilon_1, \epsilon_2 \sim \mathcal{U}[0; 0.1]$

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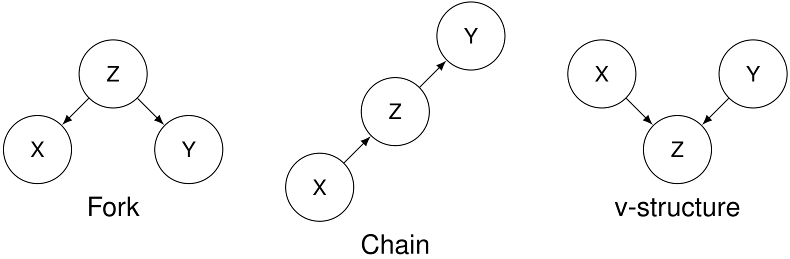
Forks, chains and v-structures



Exploiting (in)dependencies in observational data

$X, Y \sim U(-1, 1)$

$Corr(X; Y) = 0.002$

$Z = 2XY + \xi_C, \ \xi_C \sim N(0, \frac{1}{2})$

$Corr(X; Y \mid Z > 0.5) = 0.8$

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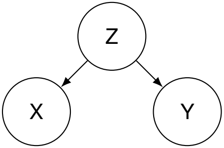
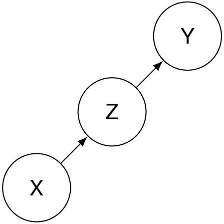
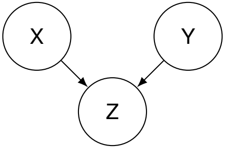


Fork



Chain



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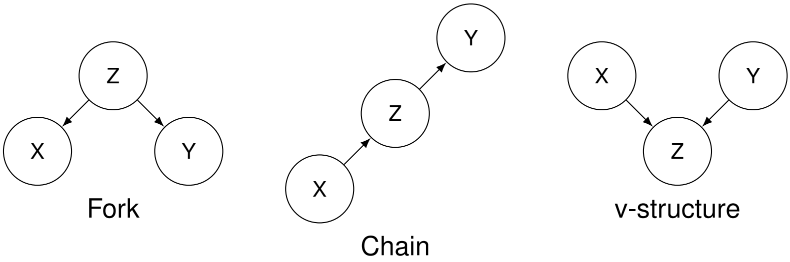

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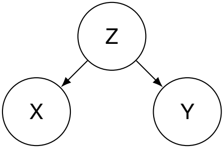

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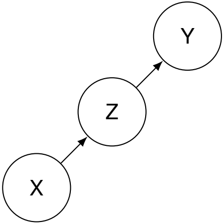

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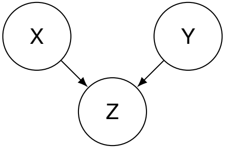

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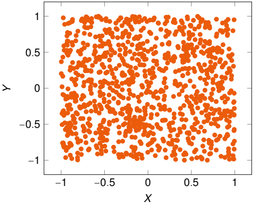

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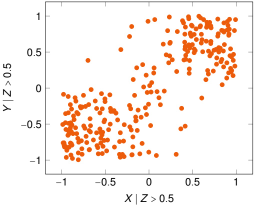

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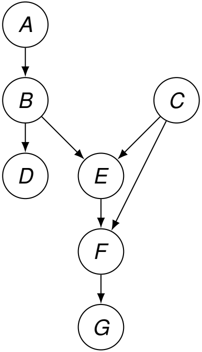
Reading conditional independencies in graphs

What conditional independencies hold in a distribution P compatible with a given graph $\mathcal{G}$?



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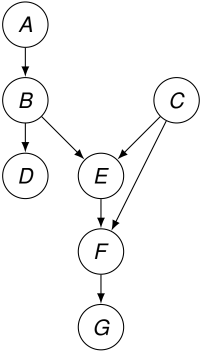

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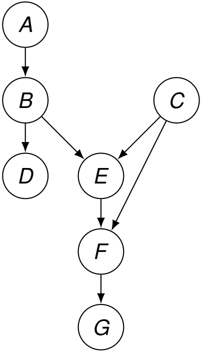

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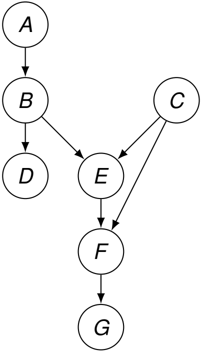

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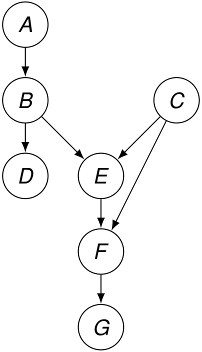

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d-Separation

Collider *A* collider is a directed graph isomorphic to $X \to Z \leftarrow Y$. We'll refer to Z in a collider as *the* collider. If the two parent vertices are not adjacent, the collider is a *v-structure* (also called *immorality*)

Active and blocked paths A path is said to be *blocked* by a set of vertices $\mathcal{Z} \in \mathcal{V}$ if:

- it contains a chain $A \to B \to C$ or a fork $A \leftarrow B \to C$ and $B \in \mathcal{Z}$, or
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A path that is not blocked is active. A path is active if every triple along the path is active, and blocked if a single triple is blocked

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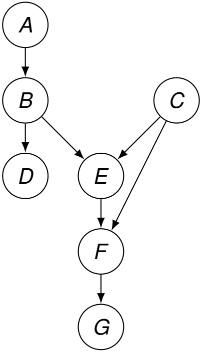

- Does $\{B\}$ block $D \leftarrow B \rightarrow E$?
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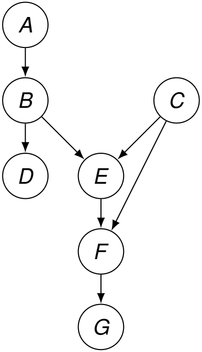

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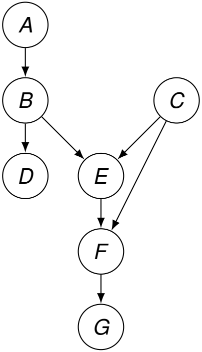

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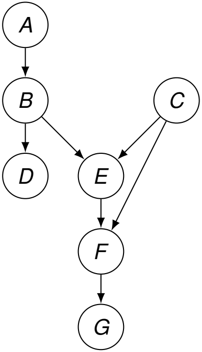

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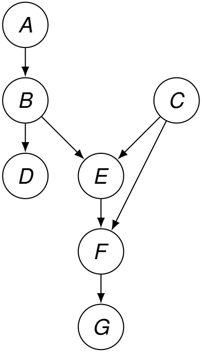

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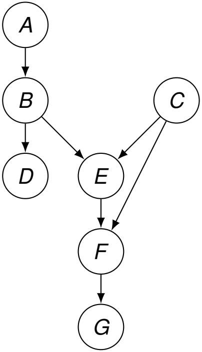

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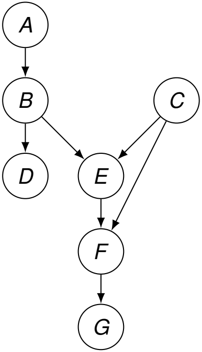

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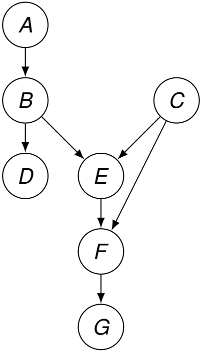

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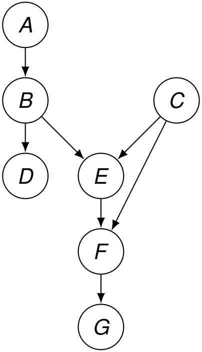

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d-separation characterizes the conditional independencies of distributions compatible with a given DAG

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(i) *Soundness* $\mathcal{X} \perp\!\!\!\perp_G \mathcal{Y} | \mathcal{Z} \Rightarrow \mathcal{X} \perp\!\!\!\perp_P \mathcal{Y} | \mathcal{Z}$ in every distribution P compatible with $\mathcal{G}$

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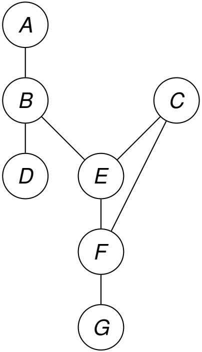

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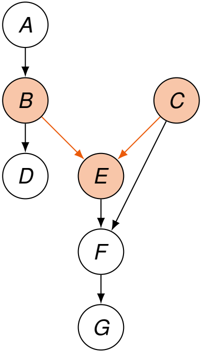

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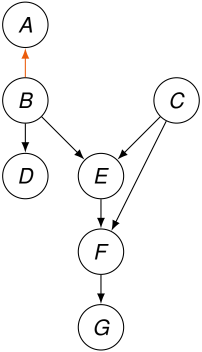

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Important lemma If X_i and X_j are not adjacent in $\mathcal{G}$, then $X_i \perp\!\!\!\perp_G X_j \mid (Pa(X_i), Pa(X_j))$ (proof on board)

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d-separation characterizes the conditional independencies of distributions compatible with a given DAG

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Implications

Theorem Given a DAG $\mathcal{G}$, $\mathcal{I}_{d-sep}(\mathcal{G}) = \cap_{p \in \mathcal{P}(G)} \mathcal{I}_{prob}(P)$

Theorem For any DAGs $\mathcal{G}_1$ and $\mathcal{G}_2$,
$\mathcal{I}_{d-sep}(\mathcal{G}_1) = \mathcal{I}_{d-sep}(\mathcal{G}_2) \Leftrightarrow \mathcal{P}(G_1) = \mathcal{P}(G_2)$

Table of content

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Conclusion

Bayesian networks, causal graphical models

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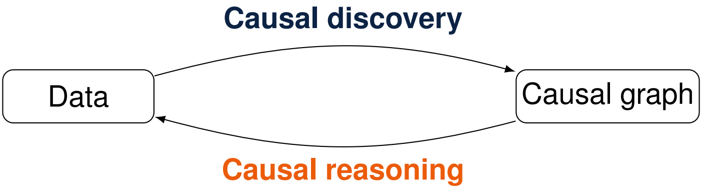

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Direct inspirations

1. *An Introduction to Causal Graphical Models*, S. Gordon (slides available at https://simons.berkeley.edu/sites/default/files/docs/18989/cau22-bcspencergordon.pdf)

2. *An Introduction to Causal Graphical Models*, V. Kumar, A. Capiln, C. Park, S. Gordon, L. Schulman (handout available at https://tinyurl.com/causalitybootcamp)

3. *Causality*, J. Pearl. Cambridge University Press, 2nd edition, 2009

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Additional readings

1. *Equivalence and Synthesis of Causal Models*, T. S. Verma, J. Pearl. Proceedings of the Sixth Annual Conference on Uncertainty in Artificial Intelligence, 1990

2. *Graphical aspects of causal models*, T. S. Verma. Technical report R-191, UCLA, 1993

3. *Probabilistic Graphical Models: Principles and Techniques*, D. Koller, N. Friedman. MIT Press, 2009

4. *Does Obesity Shorten Life? Or is it the Soda?*, J. Pearl. Journal of Causal Inference, 6(2), 2018