Counterfactual Reasoning

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Summary

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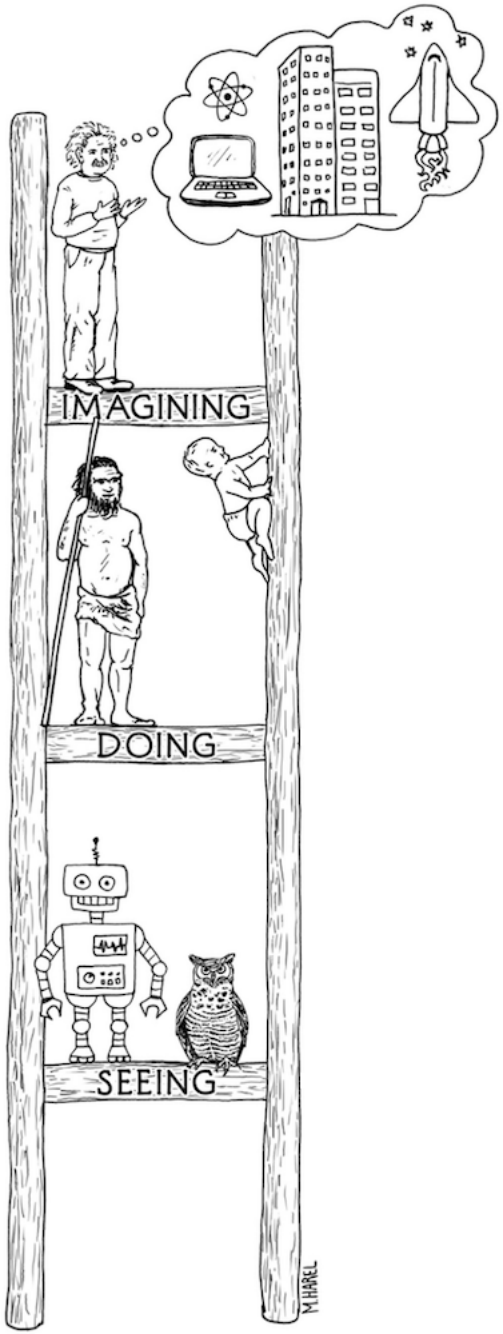
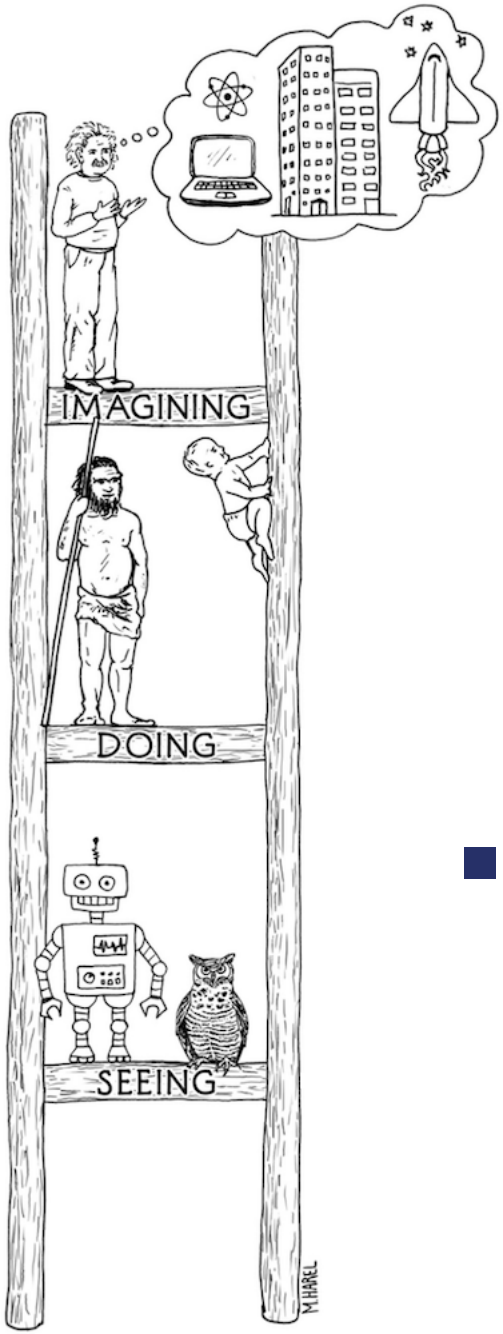
The Causal Hierarchy

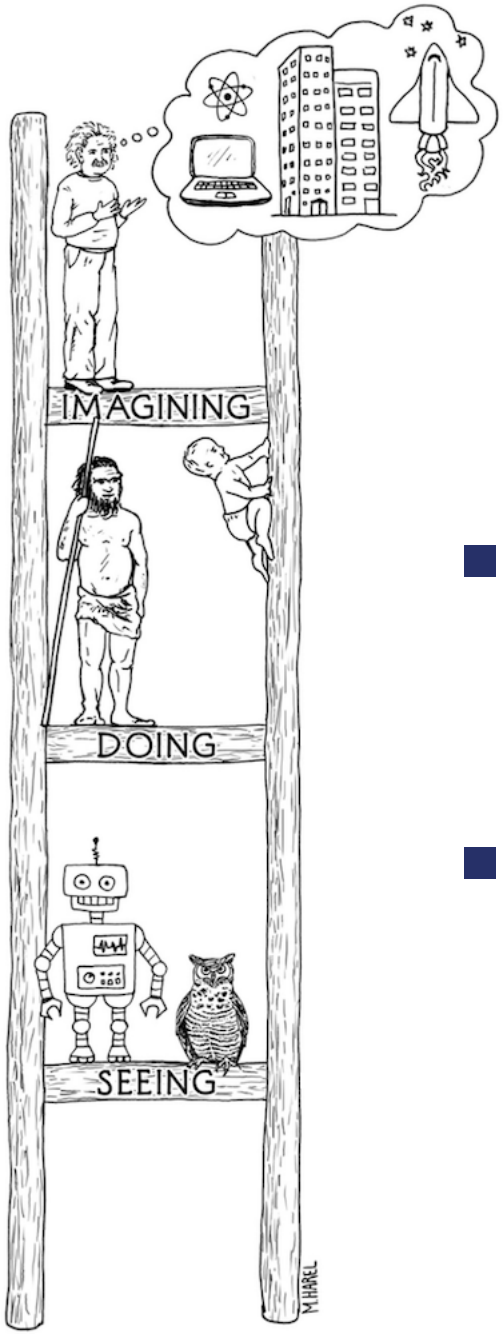


Pearl 2009



- Associations
  - What if I see…?
  - observational data
  - Tools: statistics, correlation

Pearl 2009



- Interventions
  - Questions: What if I do…? How?
  - interventional data
  - Tools: RCTs, DAGs, do-calculus
- Associations
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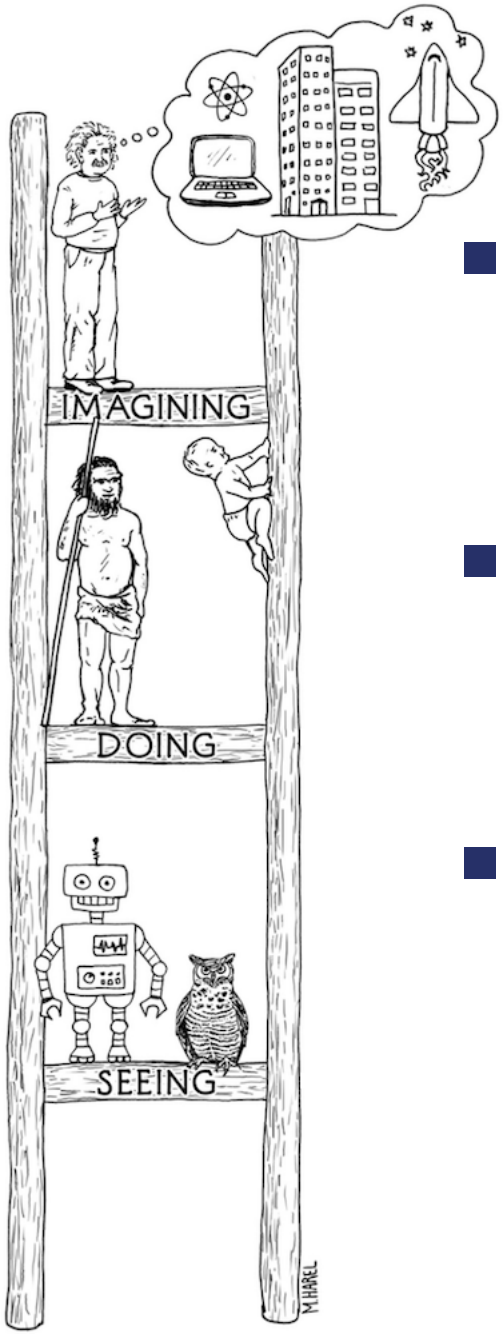

- Counterfactuals
  - Questions: What if I had done…? Why?
  - Generating process (SCM)
  - Tools: Twin Networks, SWIGs, AMWN
- Interventions
  - Questions: What if I do…? How?
  - interventional data
  - Tools: RCTs, DAGs, do-calculus
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Pearl 2009

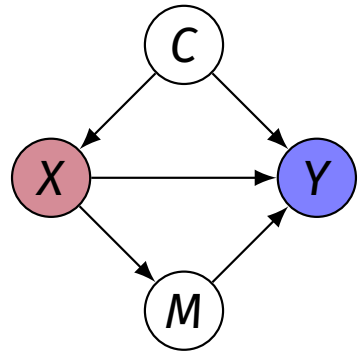
$$\xi_C, \xi_X, \xi_M, \xi_Y$$

$$C := \xi_C$$

$$X := f_X(C, \xi_X)$$

$$M := f_M(X, \xi_M)$$

$$Y := f_Y(C, X, M, \xi_Y)$$



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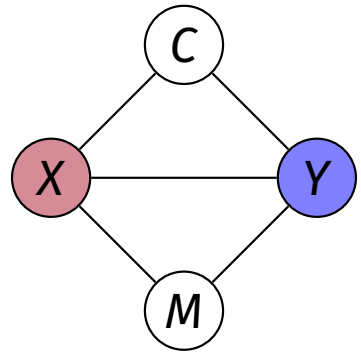

- Associations:
  $\Pr(Y = y \mid X = x)$

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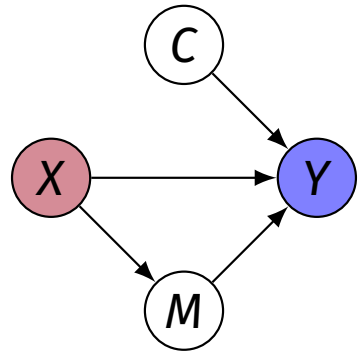

- Interventions:
  $\Pr(Y = y \mid \mathrm{do}(X = x))$
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$$\xi_C, \xi_X, \xi_M, \xi_Y$$

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$$X := x$$

$$M := f_M(x', \xi_M)$$

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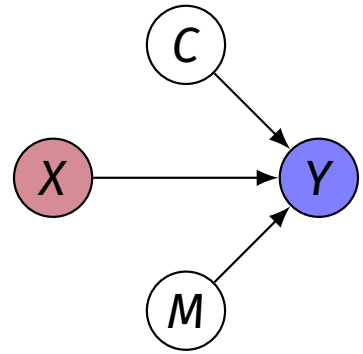

- Counterfactuals:
  $\Pr\left(Y = y \mid \text{do}\left(X = x\right), \text{do}\left(M = m_{\text{do}(X=x')}\right)\right)$
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\end{aligned}
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M &:= f_M(X_1, \xi_M) \\
X_2 &:= x \\
Y &:= f_Y(C, X_2, M, \xi_Y)
\end{aligned}
$$

Counterfactuals:

$$\Pr\left(Y = y \mid \mathrm{do}\left(X = x\right), \mathrm{do}\left(M = m_{\mathrm{do}(X=x')}\right)\right)$$

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Examples of Counterfactuals

Examples of counterfactuals

- Effect of the treatment on the treated:
  $\mathbb{E}\left(Y_{\text{do}(X=1)} \mid X = 1\right) - \mathbb{E}\left(Y_{\text{do}(X=0)} \mid X = 1\right)$

Examples of counterfactuals

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- Individual effect: "Bob did not take the treatment and died. Would he have lived had he taken the treatment?"

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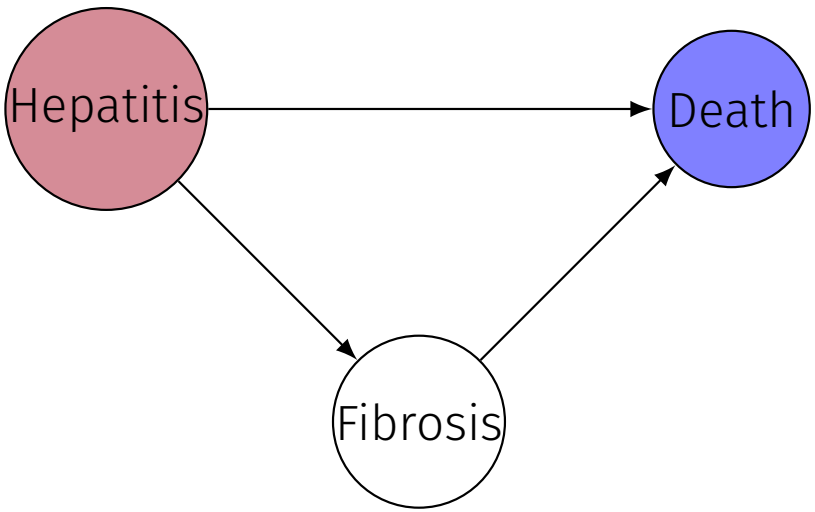
- Individual effect: "Bob did not take the treatment and died. Would he have lived had he taken the treatment?"
- Natural Direct Effect

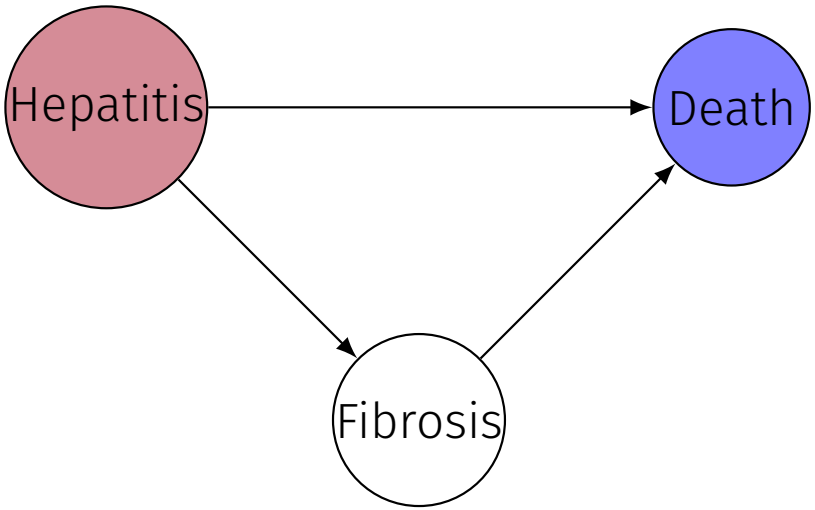
The Natural Direct Effect



Direct Effect: How will a change in the exposure (Hepatitis) modify the outcome (Death) if the mediator (Fibrosis) is kept constant?

Robins and Greenland 1992; Pearl 2001

The Natural Direct Effect



Direct Effect: How will a change in the exposure (Hepatitis) modify the outcome (Death) if the mediator (Fibrosis) is kept constant?

$$CDE(D, H_{0\to 1}, F = f) = \mathbb{E}\left(D \mid \text{do}\left(H = 1\right), \text{do}\left(F = f\right)\right) - \mathbb{E}\left(D \mid \text{do}\left(H = 0\right), \text{do}\left(F = f\right)\right)$$

This is called the controlled direct effect.

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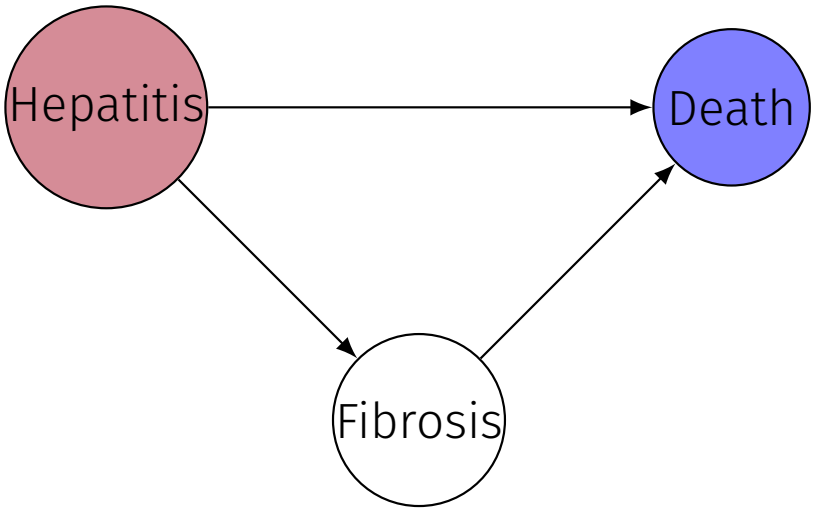
The Natural Direct Effect



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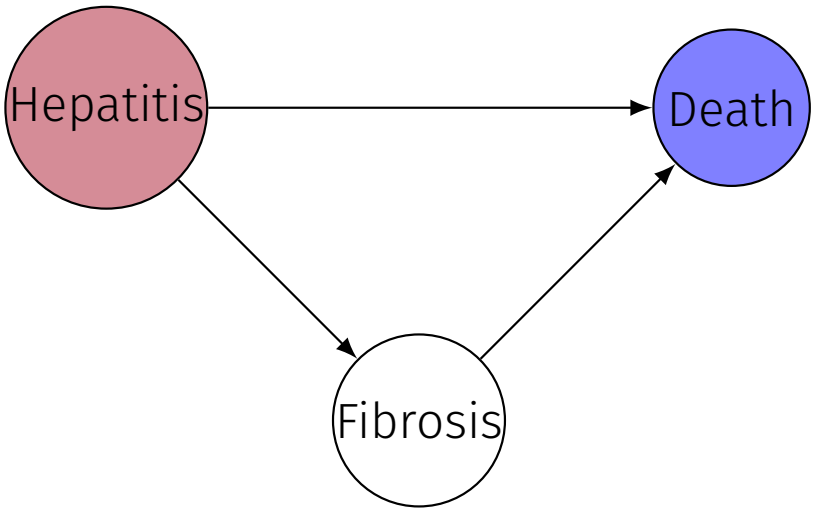
This is called the controlled direct effect.
Problem: This definition depends on the value of f. Thus there exists as many values of CDE as F has possible values.

Robins and Greenland 1992; Pearl 2001

The Natural Direct Effect



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What is the natural value for the mediator Fibrosis?

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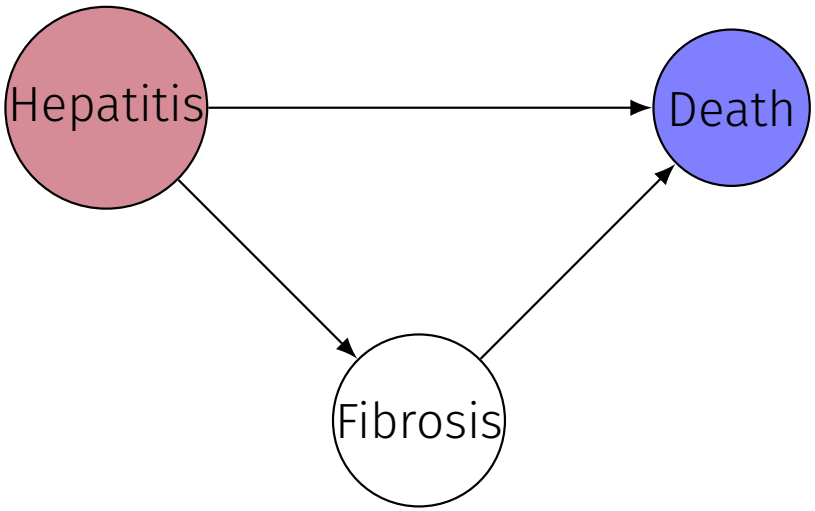

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The natural value for Fibrosis is the value it would have taken had we not modified the exposure (Hepatitis)

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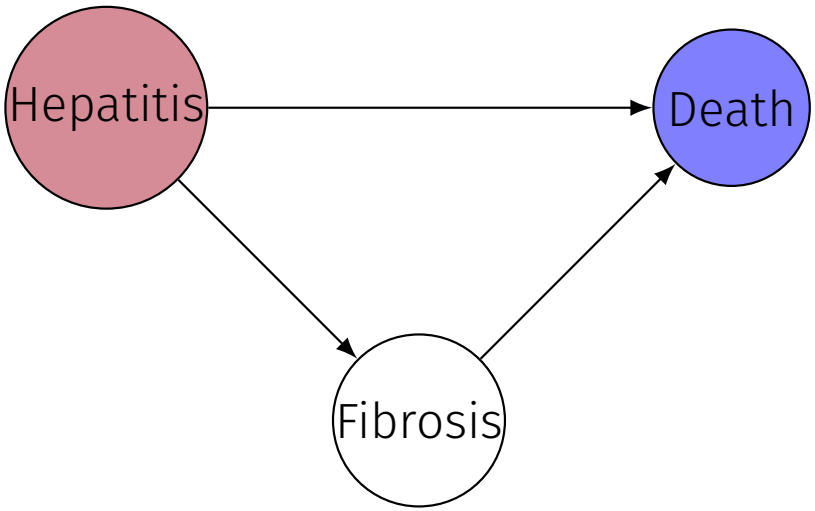

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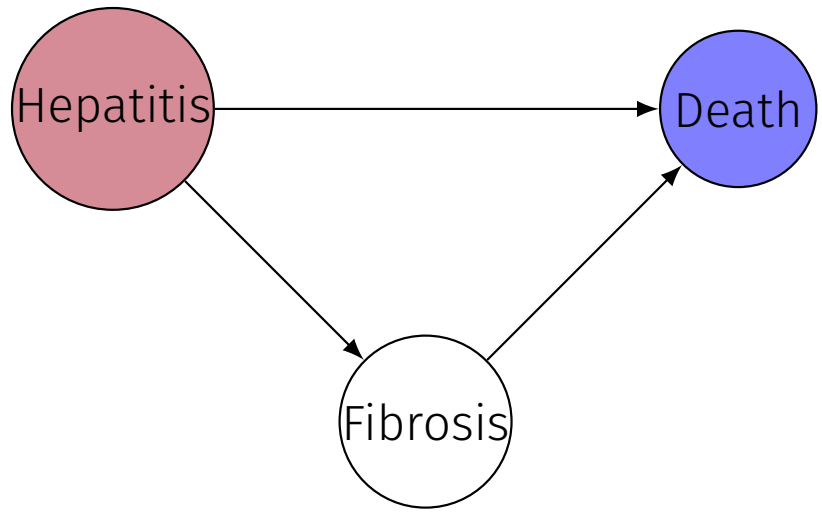
$$NDE(D, H_{0\to 1}, F) = \mathbb{E}\left(D \mid \text{do}\left(H = 1\right), \text{do}\left(F = f_{\text{do}(H=0)}\right)\right) - \mathbb{E}\left(D \mid \text{do}\left(H = 0\right), \text{do}\left(F = f_{\text{do}(H=0)}\right)\right)$$

Robins and Greenland 1992; Pearl 2001

Toy Example

$$\xi_H, \xi_F, \xi_D \leftarrow \mathcal{U}(0,1)$$
$$H := \xi_H$$
$$F := H \vee \xi_F$$
$$D := (H \vee F) \wedge \xi_D$$



Toy Example

observational Data:

Noise			Variables		
ξ_H	ξ_F	ξ_D	H	F	D
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	1	1	1
1	1	0	1	1	0
1	1	1	1	1	1

$$\mathbb{E}(D \mid H = 1, F = 0) - \mathbb{E}(D \mid H = 0, F = 0)$$
$$= 0 - 0 = 0$$

Toy Example

interventional Data:

Noise			do $(H = 0, F = 0)$			do $(H = 1, F = 0)$		
ξ_H	ξ_F	ξ_D	H	F	D	H	F	D
0	0	0	0	0	0	1	0	0
0	0	1	0	0	0	1	0	1
0	1	0	0	0	0	1	0	0
0	1	1	0	0	0	1	0	1
1	0	0	0	0	0	1	0	0
1	0	1	0	0	0	1	0	1
1	1	0	0	0	0	1	0	0
1	1	1	0	0	0	1	0	1

Controlled direct effect (CDE)

$$\mathbb{E}(D \mid \text{do}(H = 1, F = 0)) - \mathbb{E}(D \mid \text{do}(H = 0, F = 0))$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$

Toy Example

Couterfactual Data:

Noise			$\text{do}\left(H=0, F=f_{\text{do}(H=0)}\right)$			$\text{do}\left(H=1, F=f_{\text{do}(H=0)}\right)$		
ξ_H	ξ_F	ξ_D	H	F	D	H	F	D
0	0	0	0	0	0	1	0	0
0	0	1	0	0	0	1	0	1
0	1	0	0	1	0	1	1	0
0	1	1	0	1	1	1	1	1
1	0	0	0	0	0	1	0	0
1	0	1	0	0	0	1	0	1
1	1	0	0	1	0	1	1	0
1	1	1	0	1	1	1	1	1

Natural direct effect:

$$\mathbb{E}\left(D \mid \text{do}\left(H=1, F=f_{\text{do}(H=0)}\right)\right) - \mathbb{E}\left(D \mid \text{do}\left(H=0, F=f_{\text{do}(H=0)}\right)\right)$$
$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

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Twin Networks

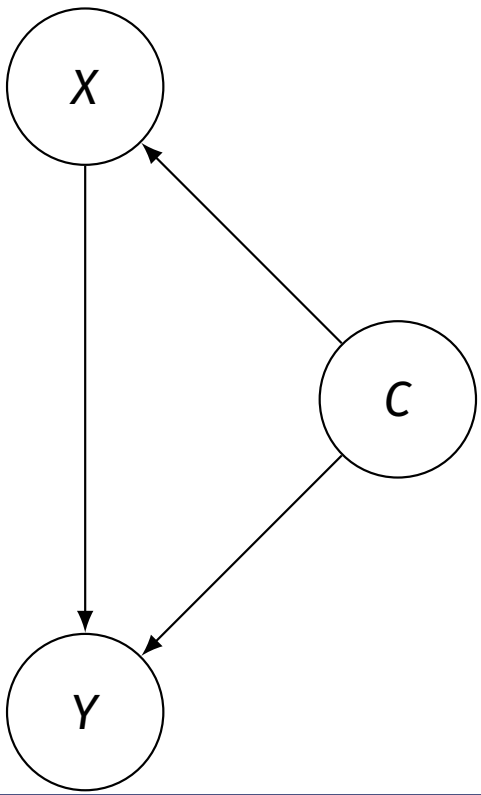
Twin Networks

- One graph for the actual world

Balke and Pearl 1994

Twin Networks

- One graph for the actual world



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- One graph for the actual world
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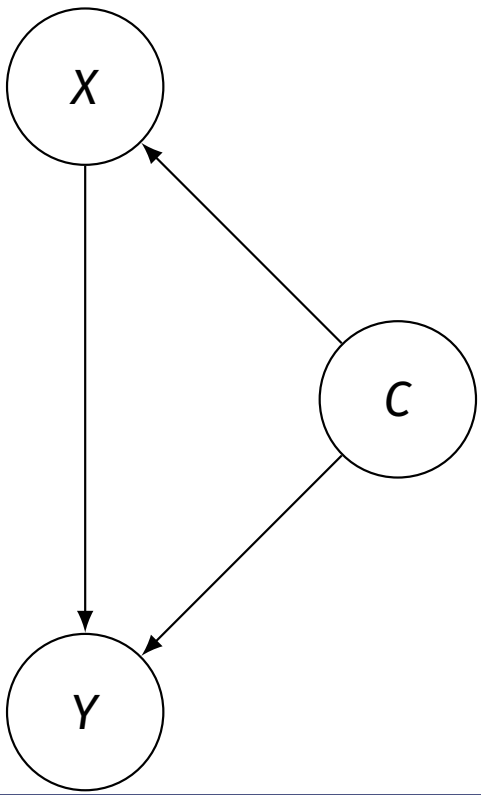

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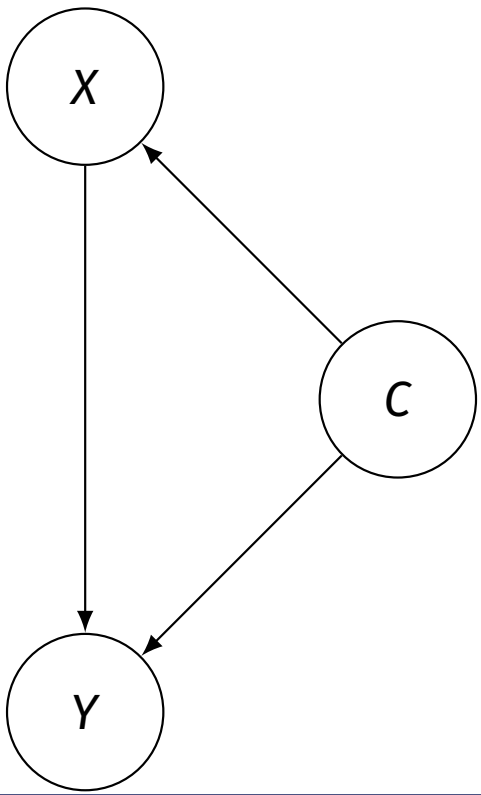

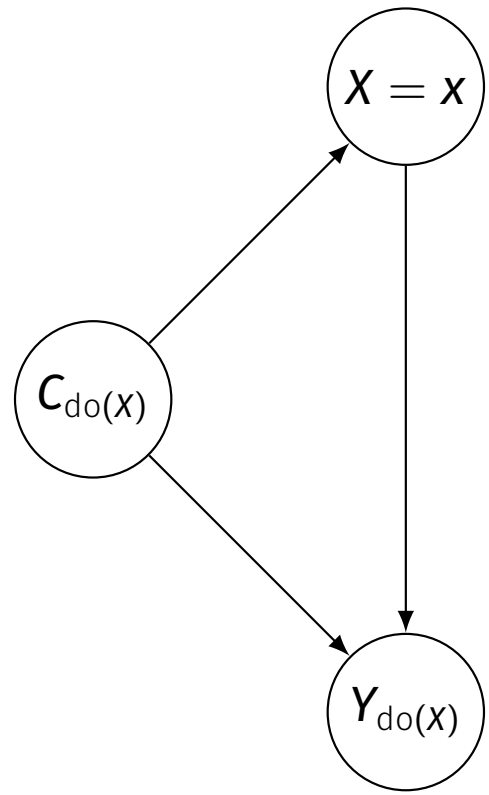

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- One graph for the actual world
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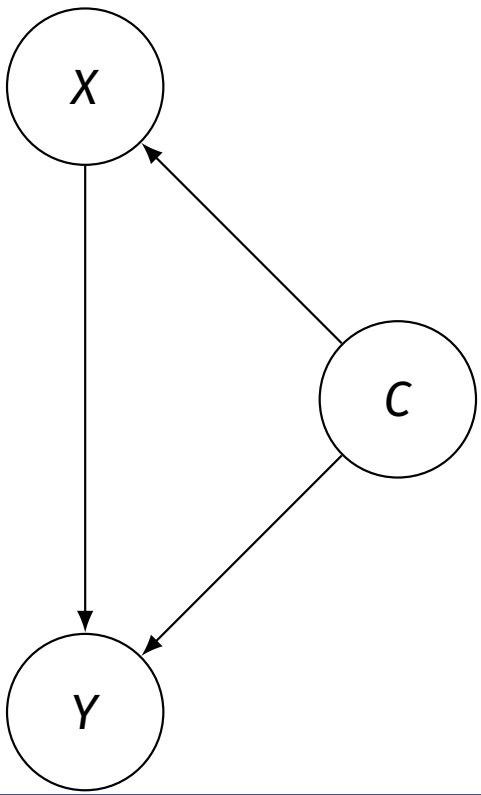
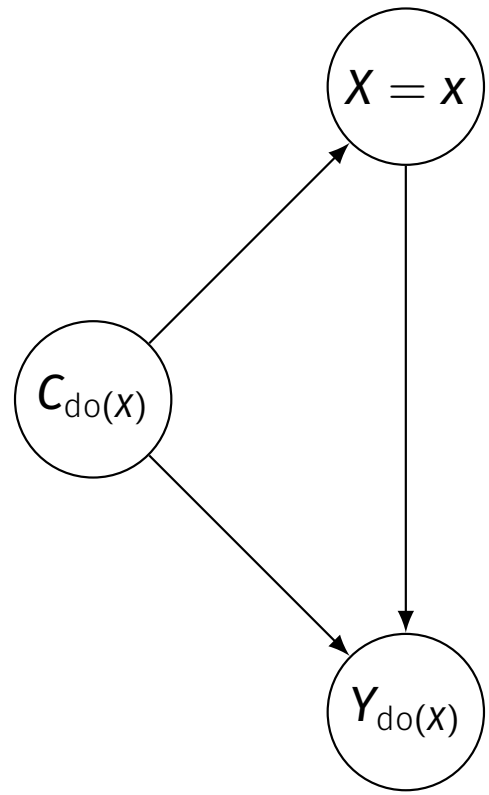
- Common noise variables





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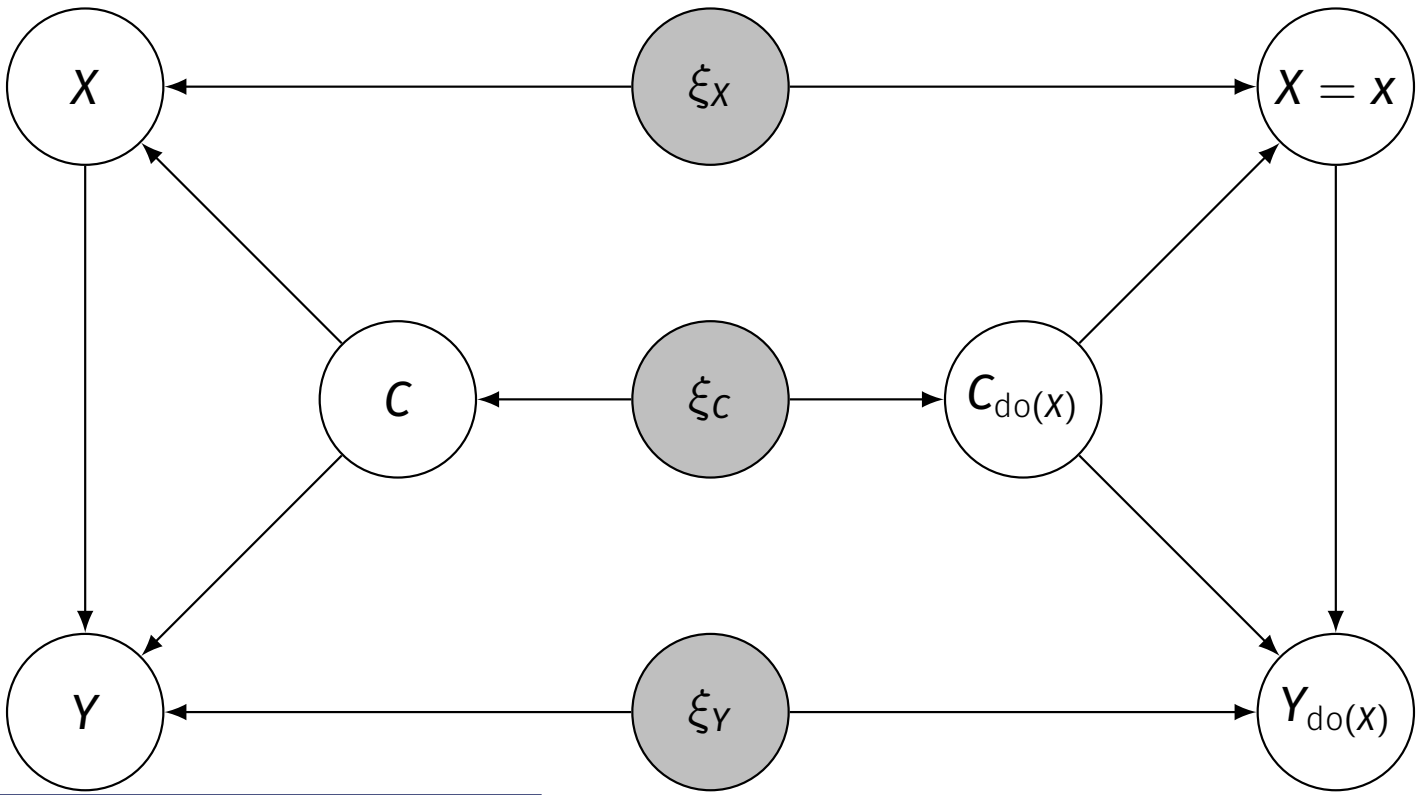

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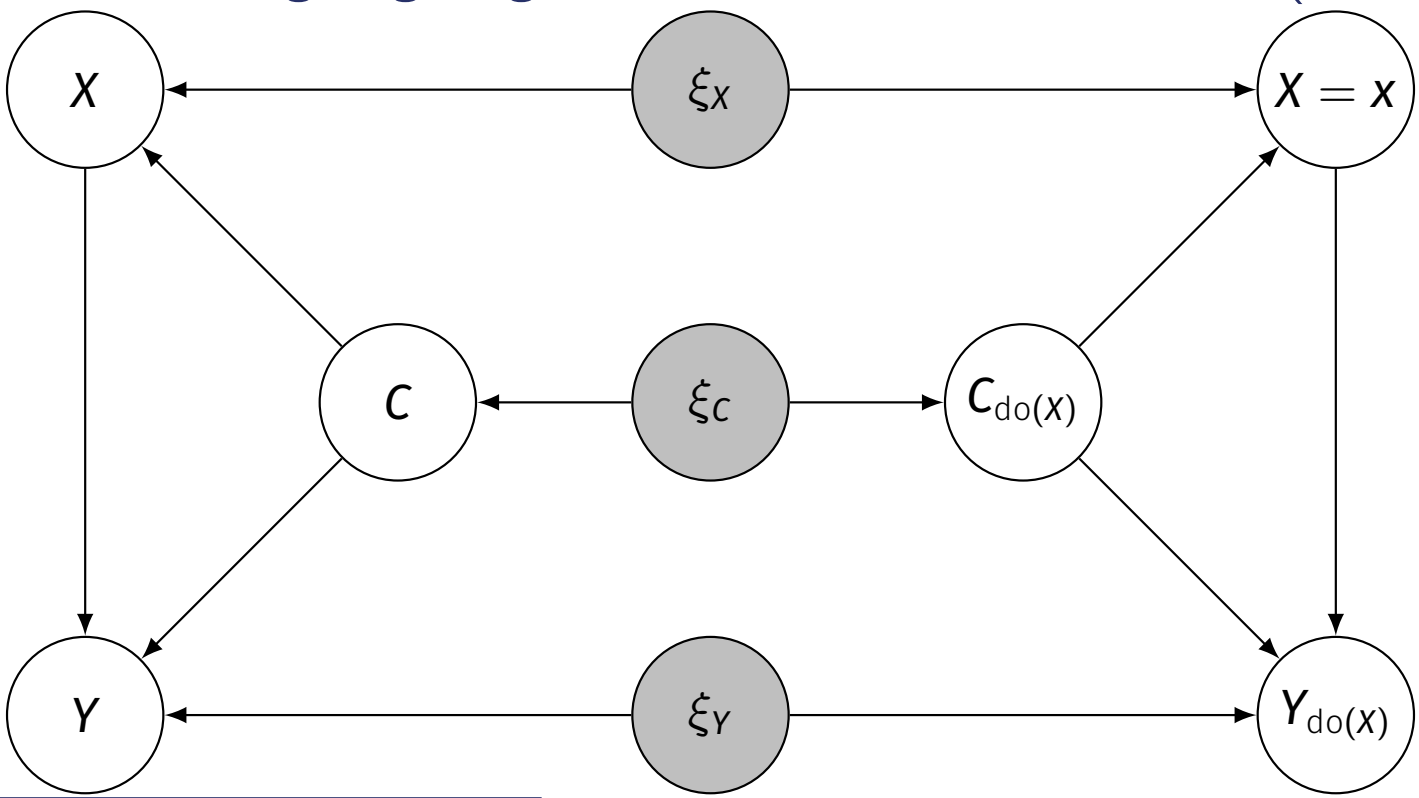
- Common noise variables
- Remove edges going in the intervened node (do $(X = x)$)



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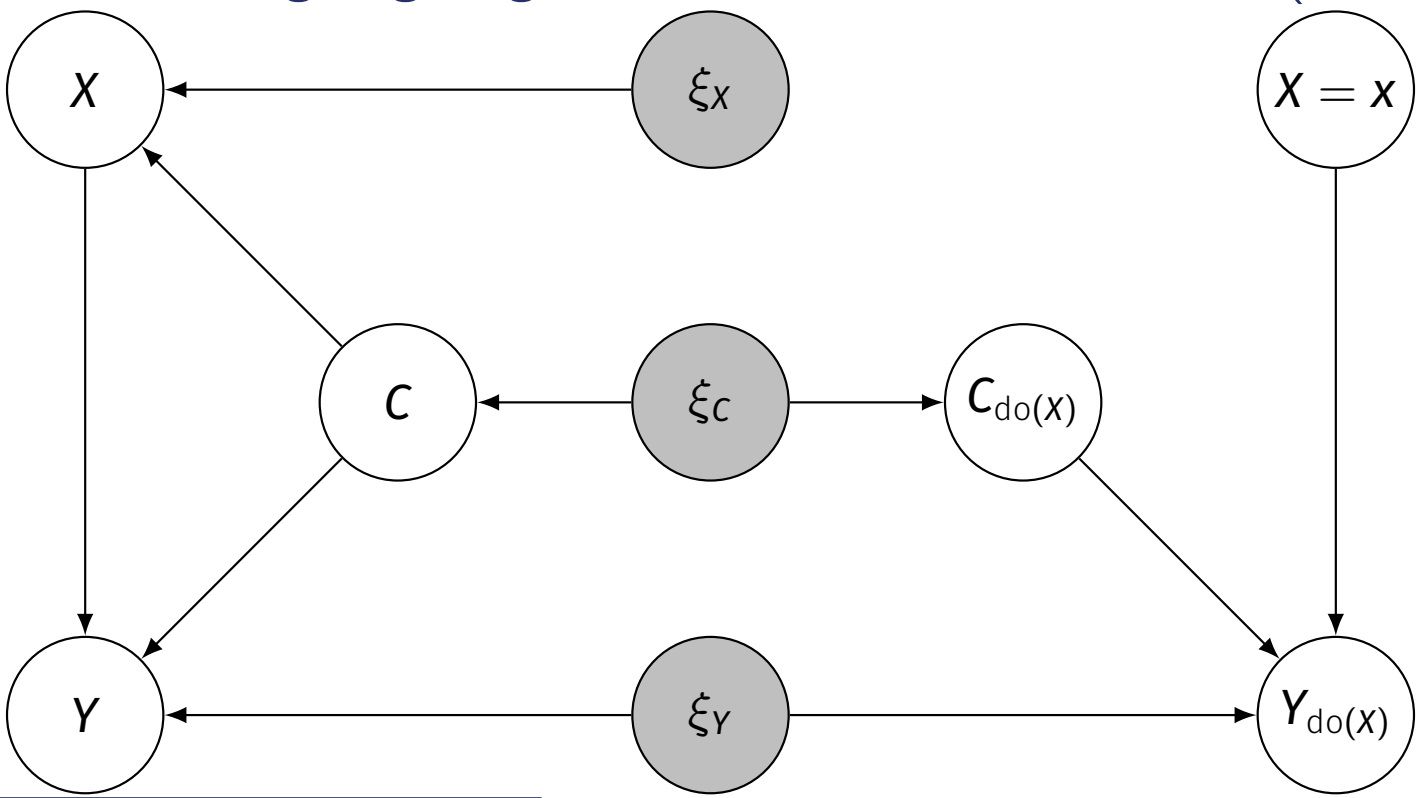

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Twin Networks

What can Twin Networks be used for?

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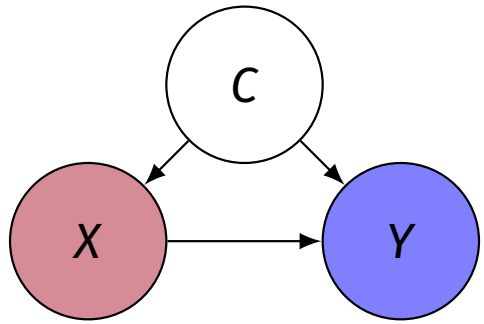
SCM framework (Pearl)

- Assume the graph
- Backdoor criterion: C blocks all backdoor paths from X to Y.



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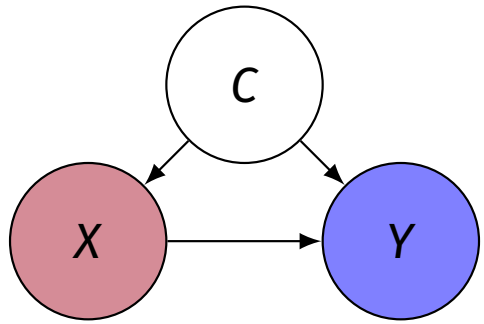

Potential Outcome framework (Robins)

- Assume conditional exchangeability

$$Y_{do(X)} \perp\!\!\!\perp_{Pr} X \mid C$$

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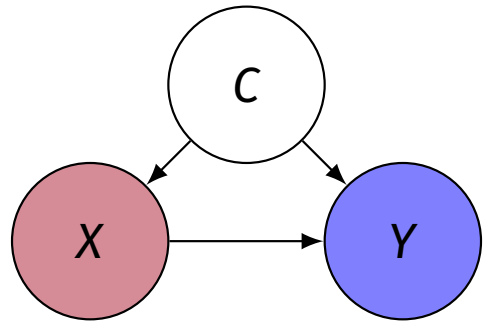


Potential Outcome framework (Robins)

- Assume conditional exchangeability

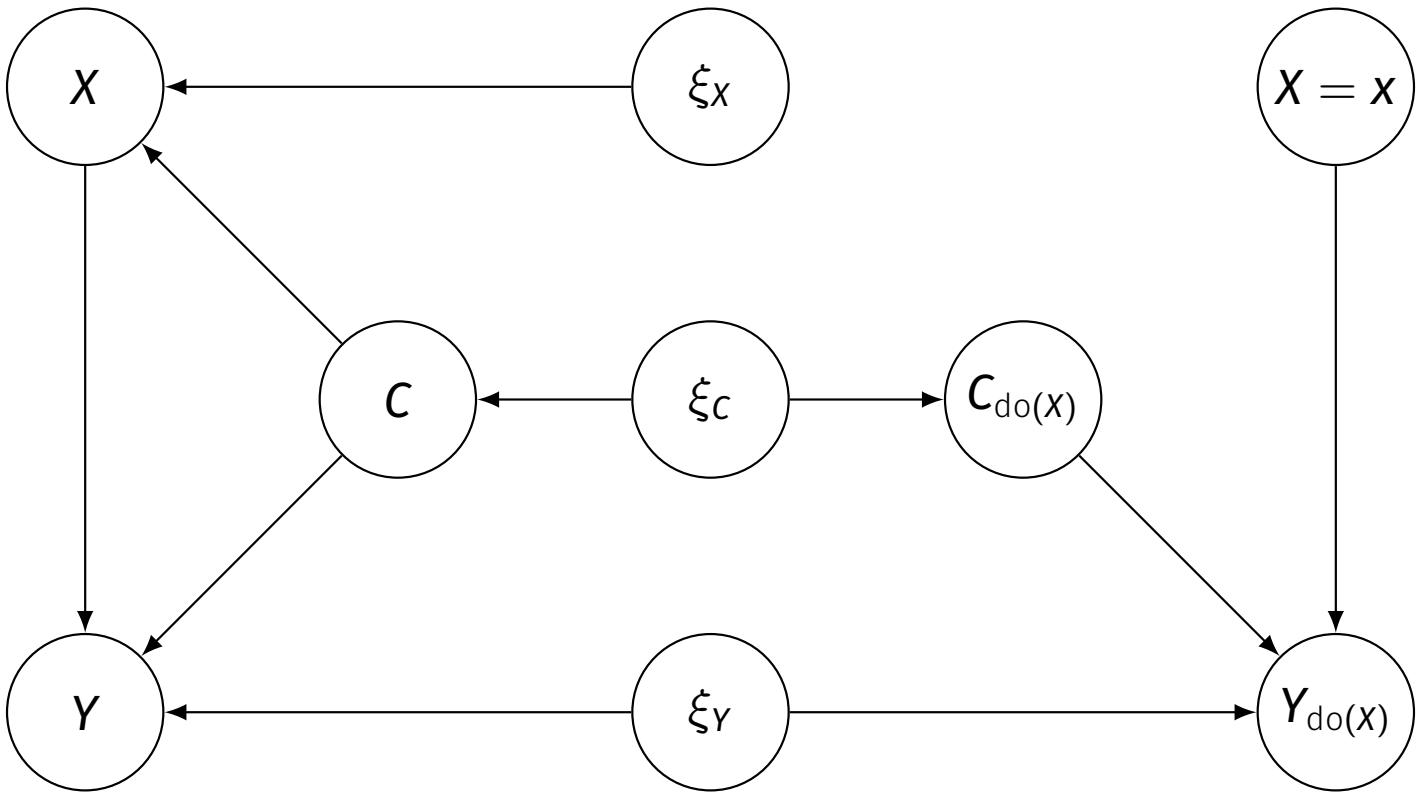
$$Y_{do(X)} \perp\!\!\!\perp_{Pr} X \mid C$$

Average Treatment Effect:

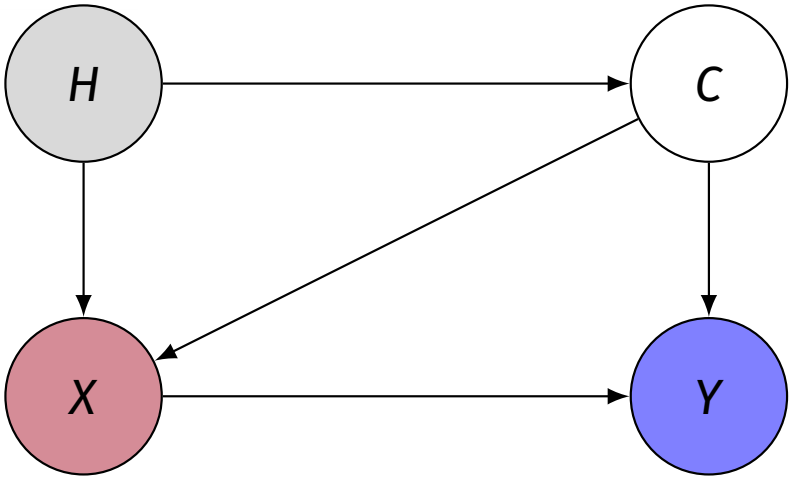
$$\sum_c \left(\mathbb{E}\left(Y \mid X = 1, C = c\right) - \mathbb{E}\left(Y \mid X = 0, C = c\right)\right) \Pr\left(C = c\right)$$

Twin Networks

Conditional Exchangeability: $Y_{do(X)} \perp\!\!\!\perp_d X \mid C$?



Twin Networks: an example of non-completeness



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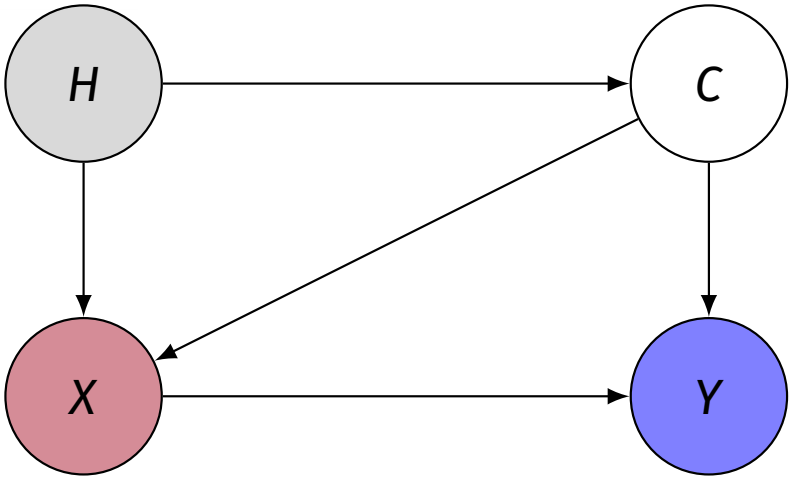

Does C verify the backdoor criterion for X,Y?

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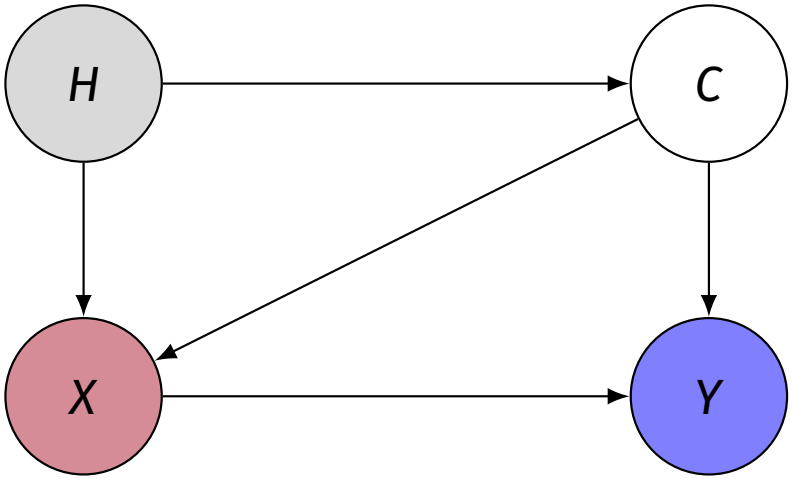

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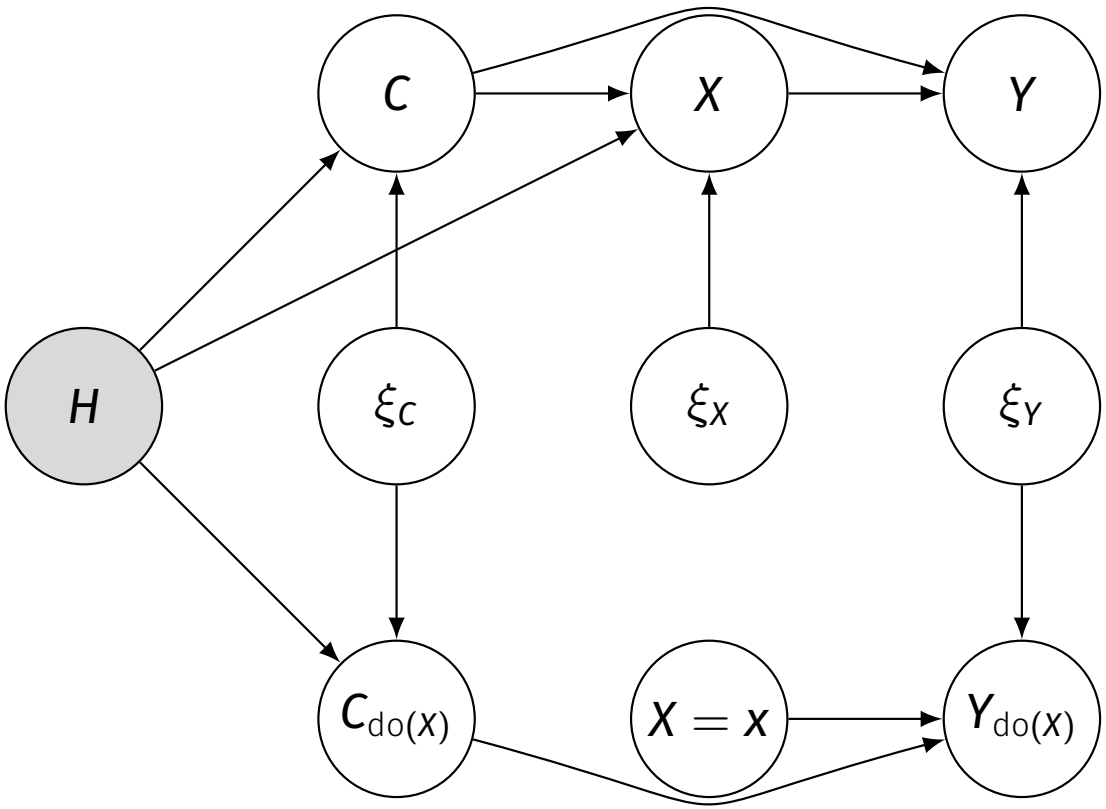
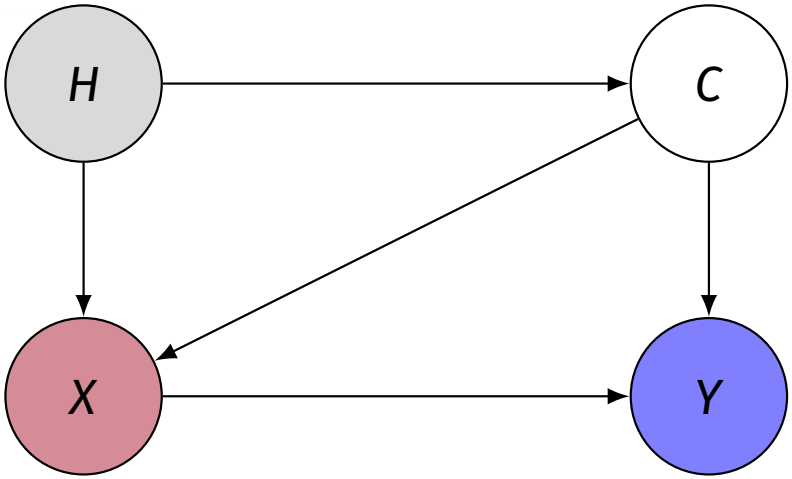
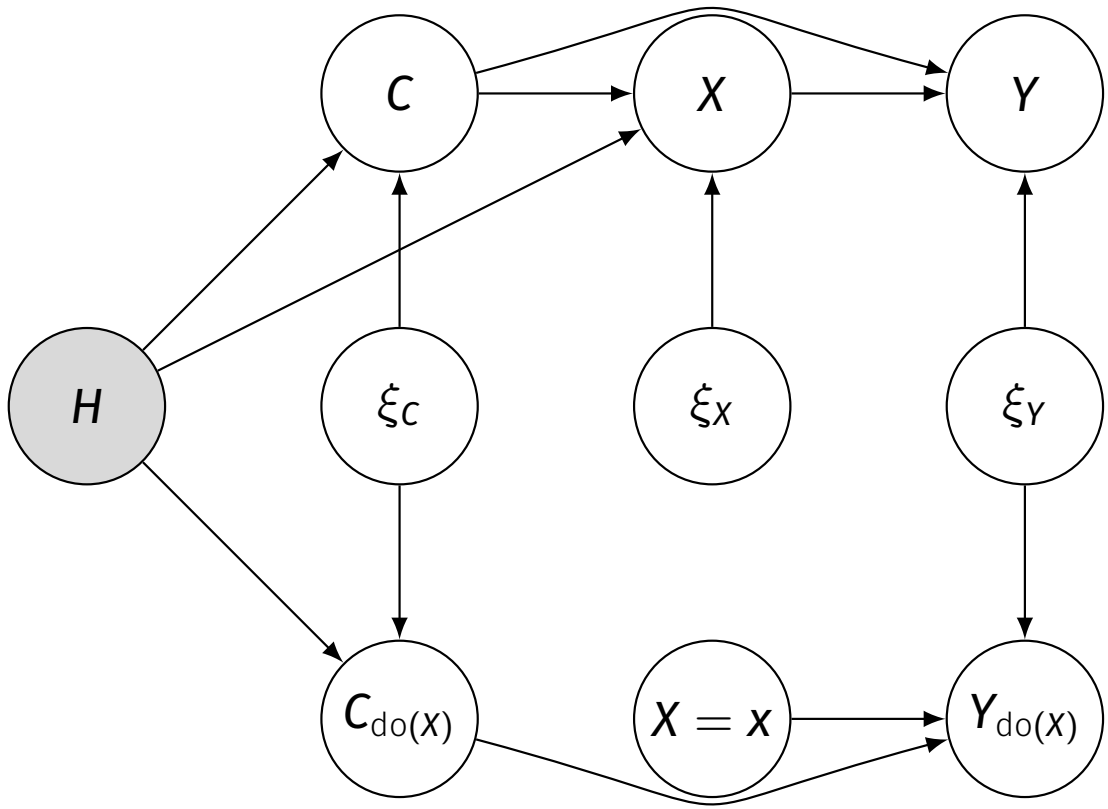


Twin Networks: an example of non-completeness



Does C verify the backdoor criterion for X,Y?

$Y_{do(X)} \perp\!\!\!\perp_d X \mid C$?



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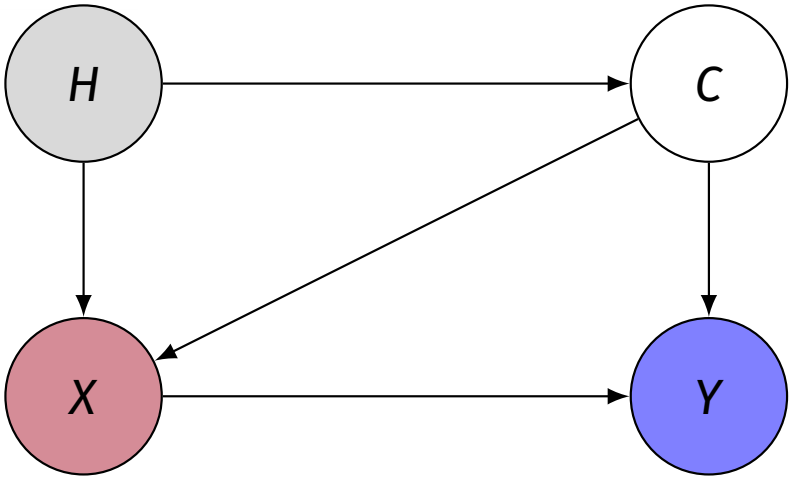

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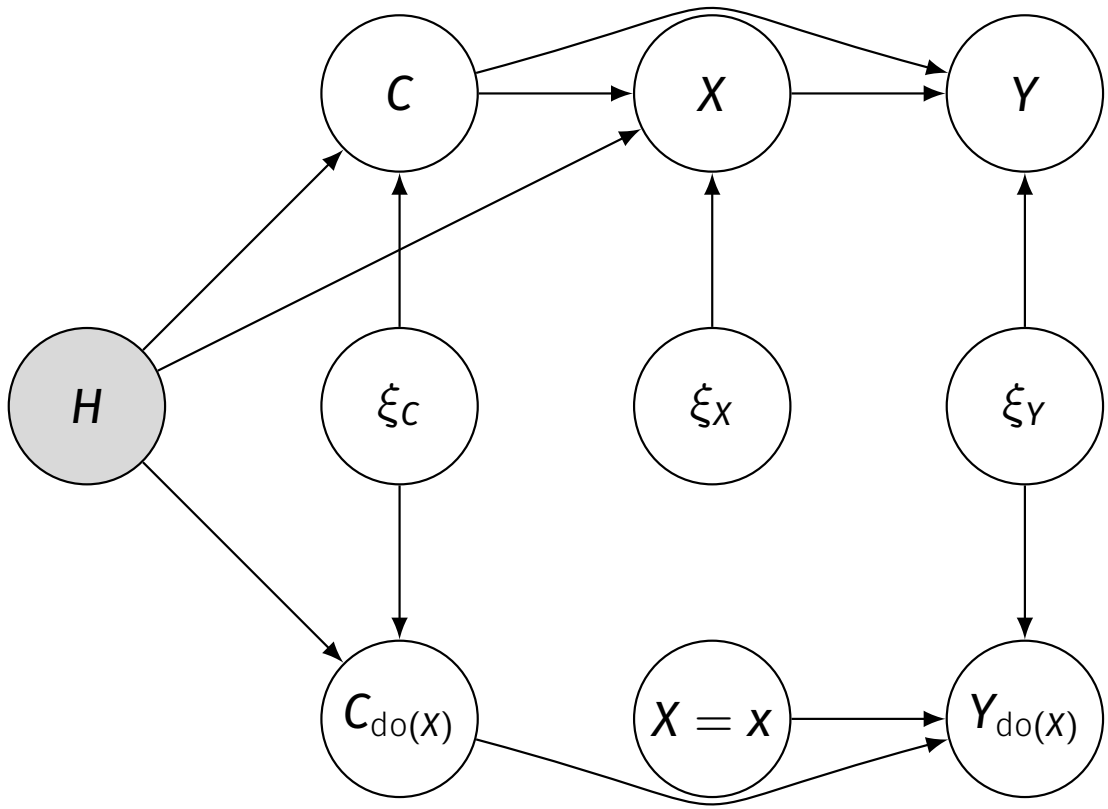

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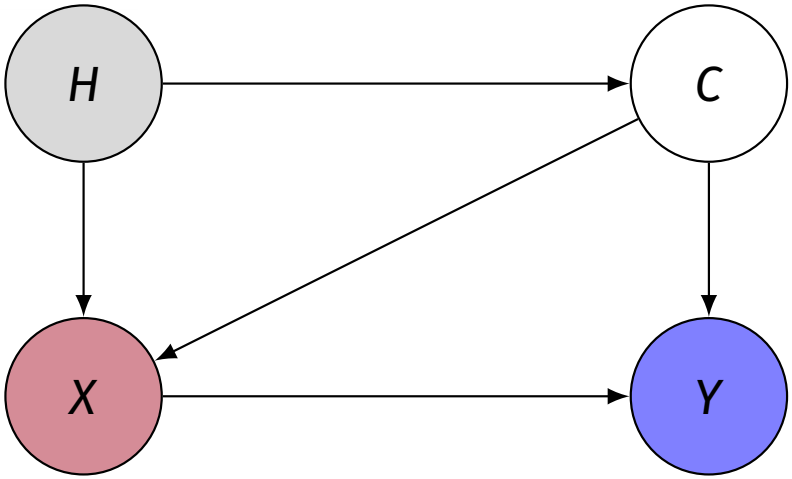

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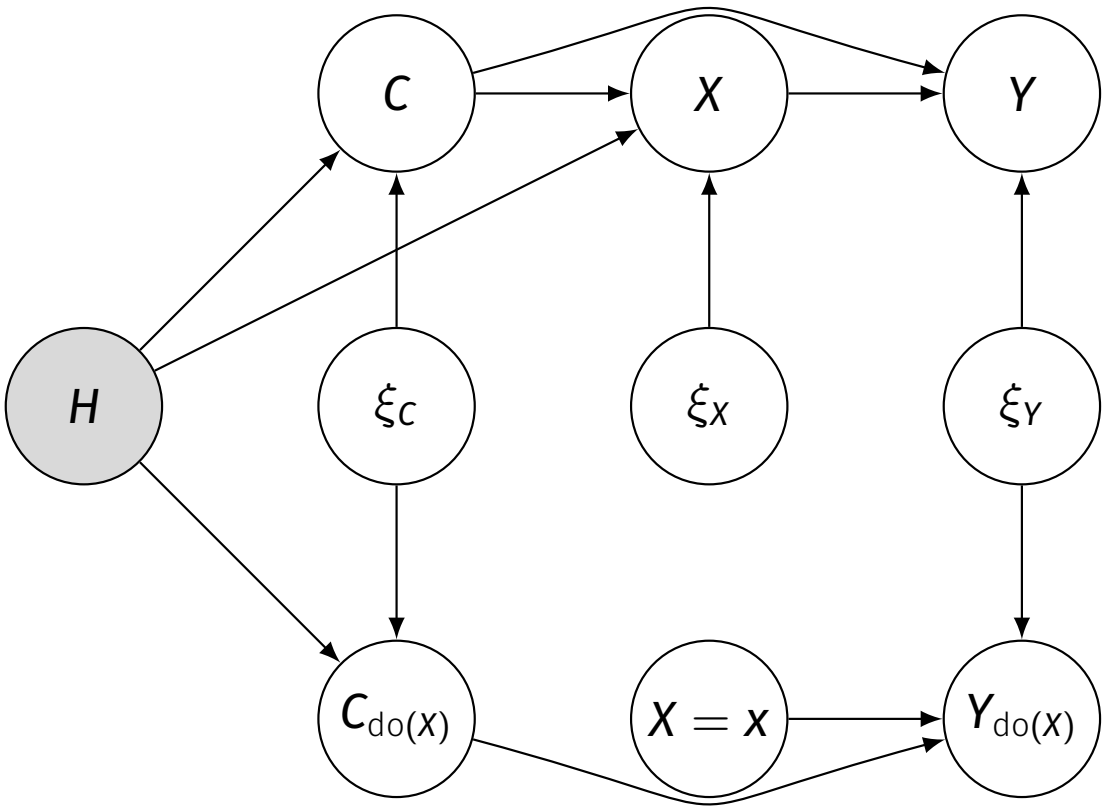
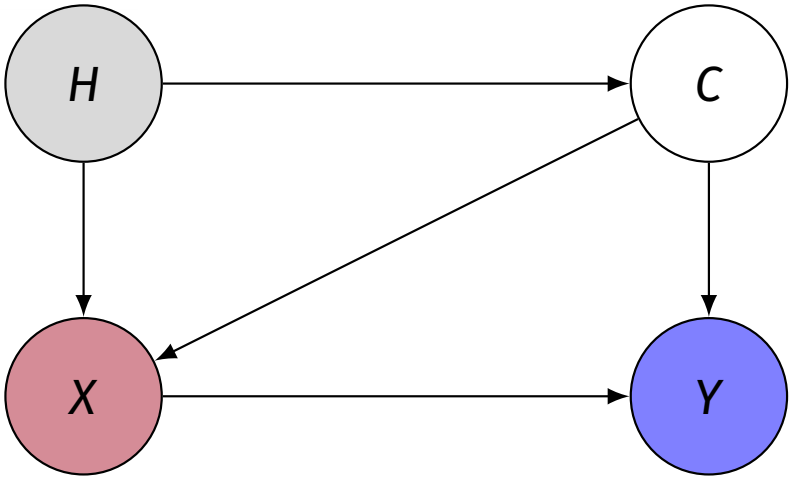


Twin Networks: an example of non-completeness



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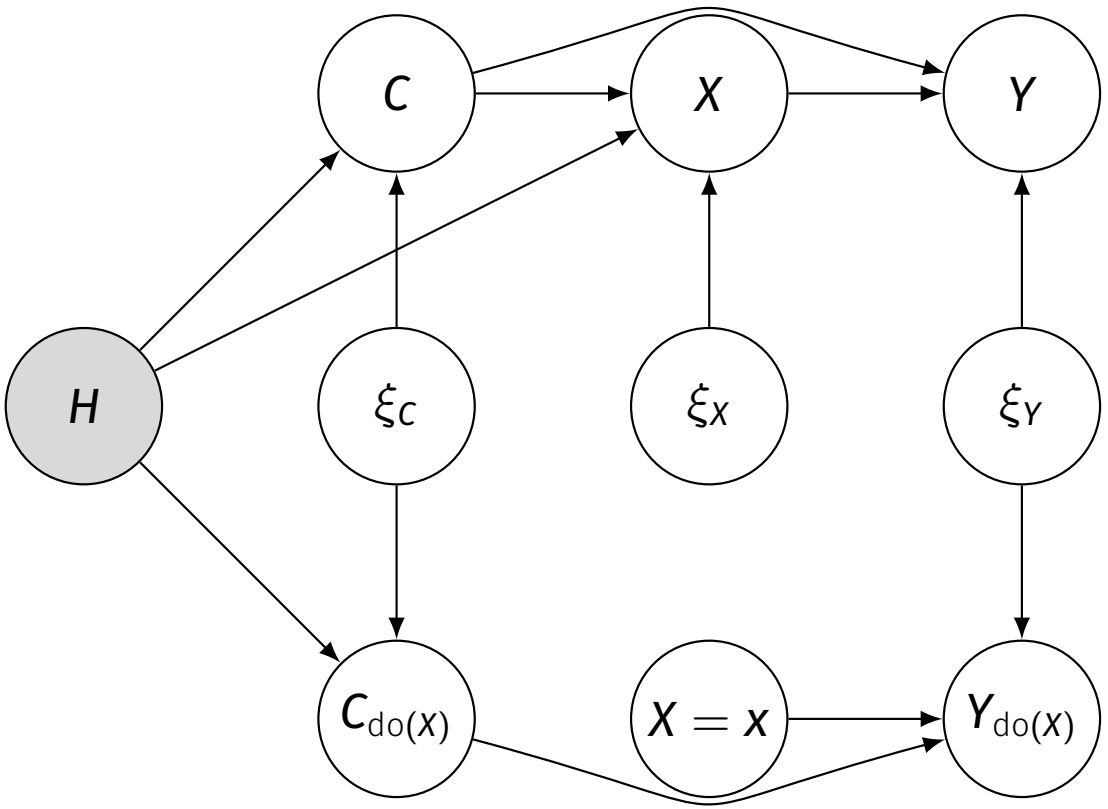
Yes!



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4

Single World Intervention Graphs (SWIGs)

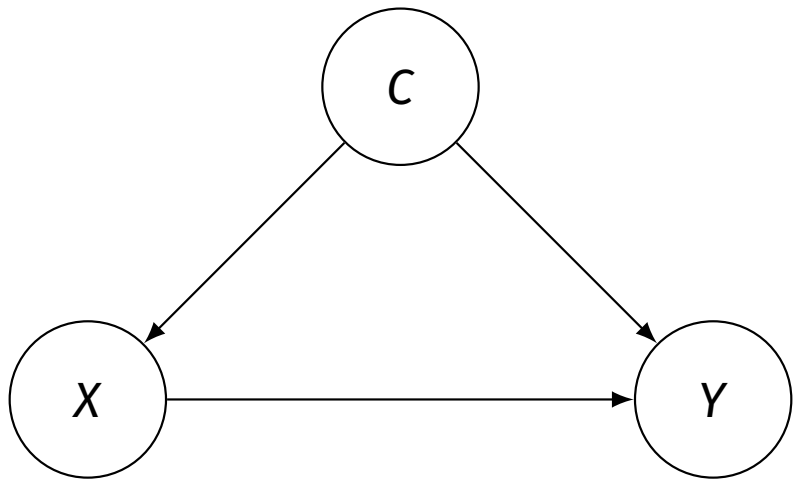
SWIGs

- One graph for both worlds

Richardson and Robins 2013

SWIGs

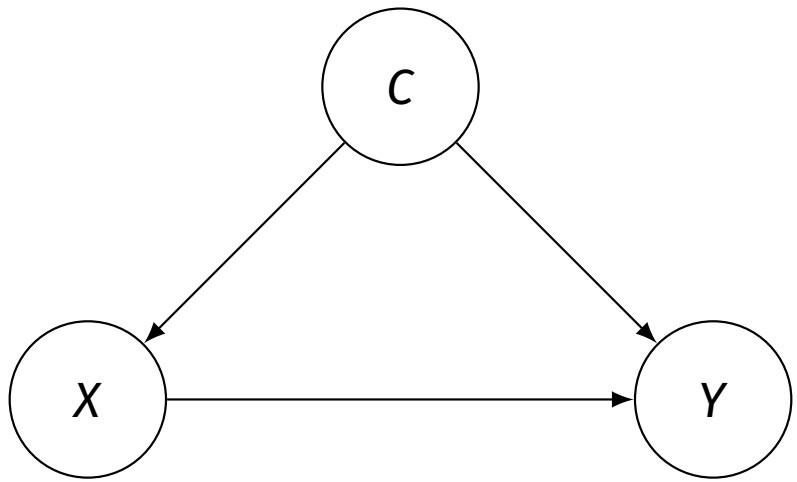
- One graph for both worlds



Richardson and Robins 2013

SWIGs

- One graph for both worlds
- Divide the intervened nodes (do $(X = x)$)



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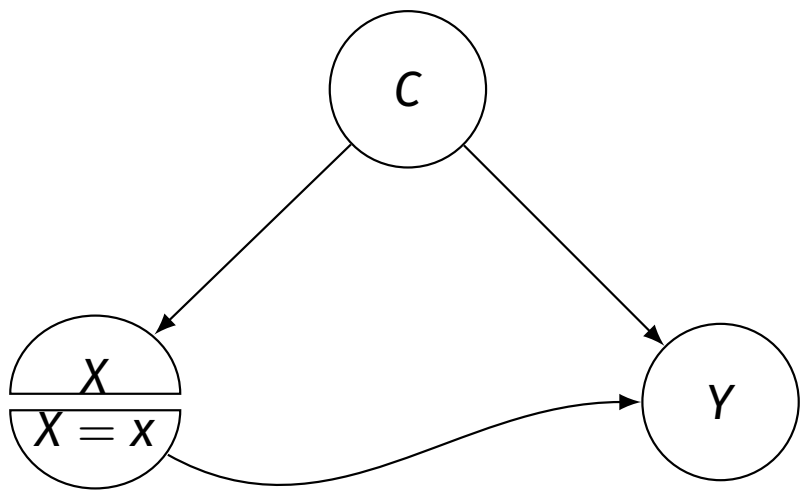
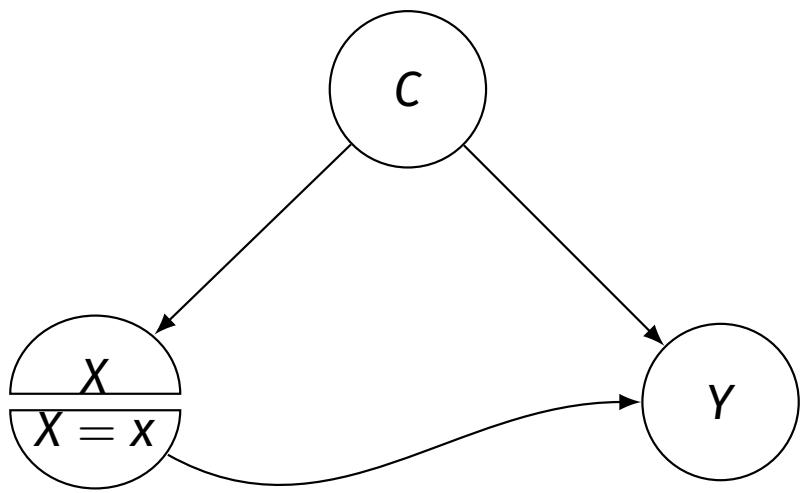


Richardson and Robins 2013

SWIGs

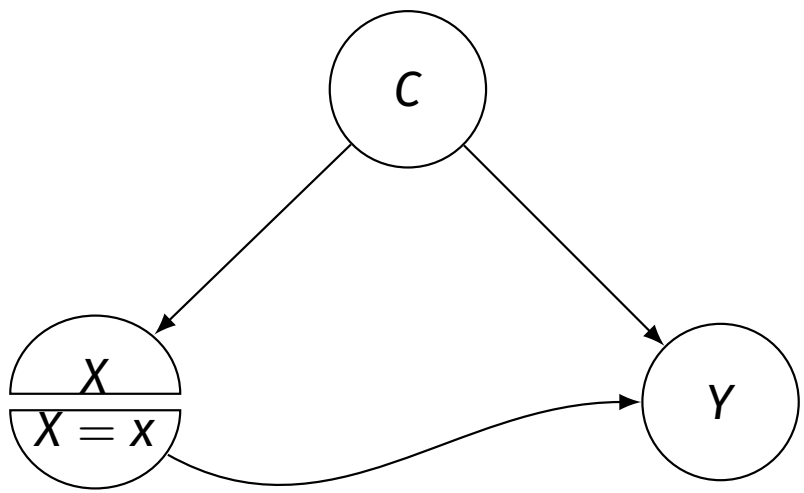
- One graph for both worlds
- Divide the intervened nodes (do $(X = x)$)
- Notice the edges



Richardson and Robins 2013

SWIGs

- One graph for both worlds
- Divide the intervened nodes (do $(X = x)$)
- Notice the edges
- Relabel nodes descendant of interventions



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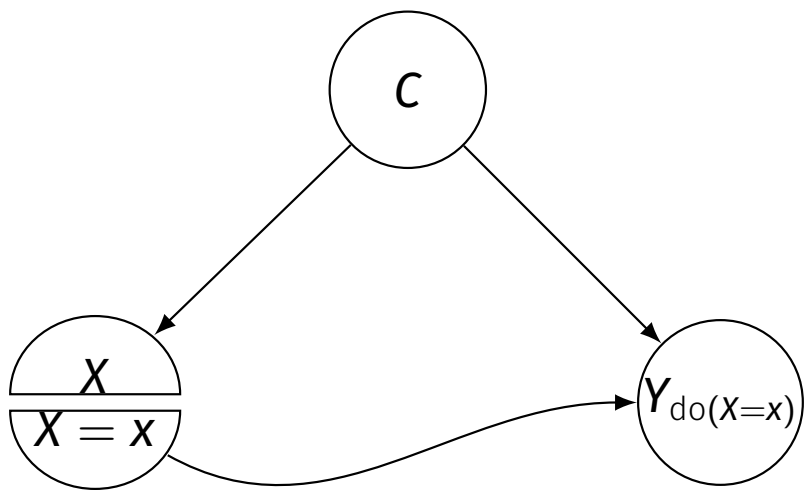
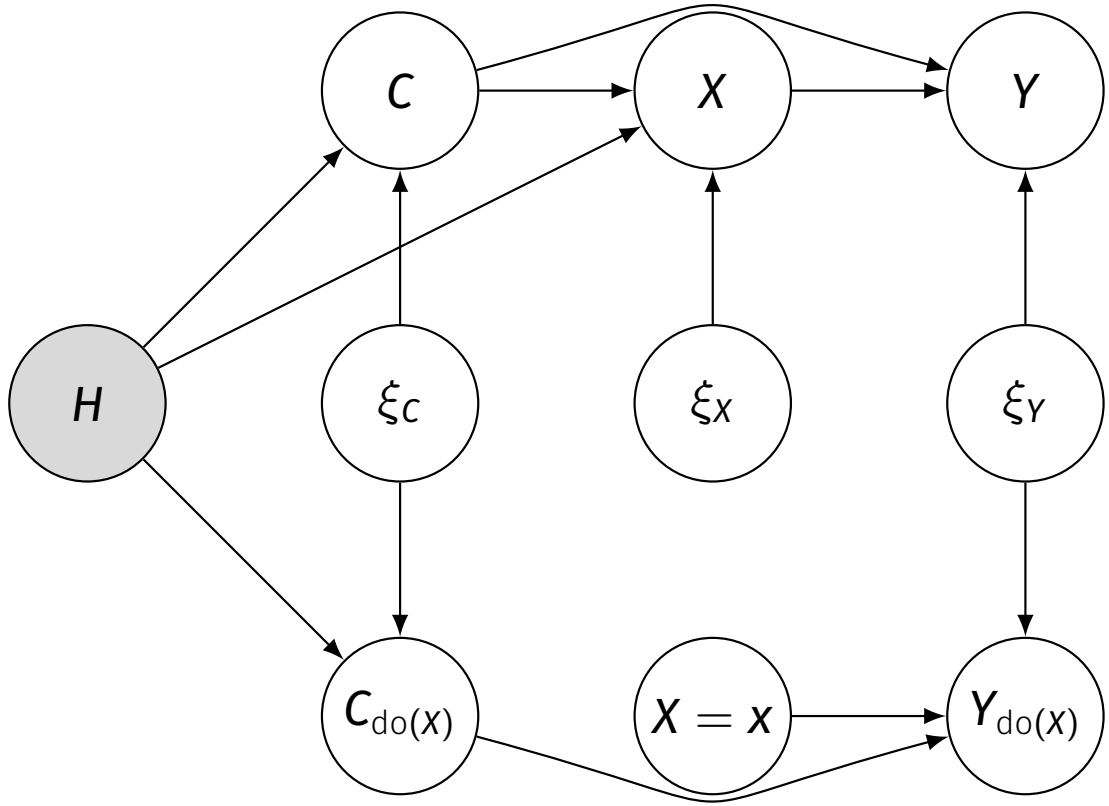
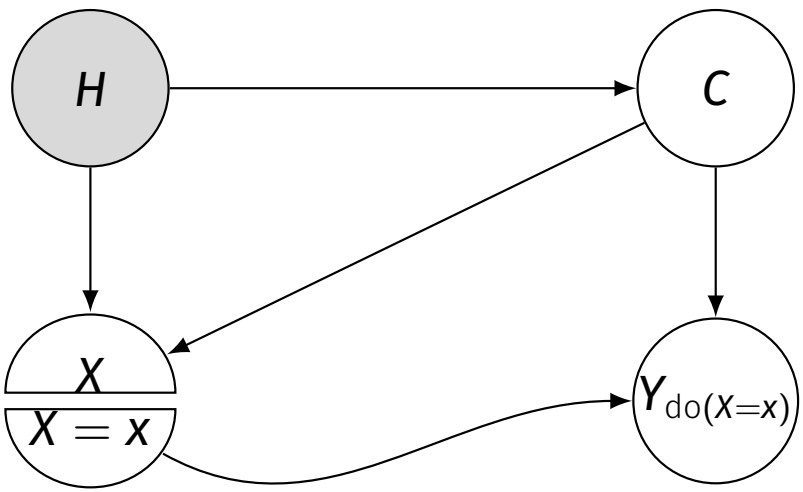


Richardson and Robins 2013

SWIGs





$Y_{do(x)} \perp\!\!\!\perp_d X \mid C$?

DO-CALCULUS

The do-calculus is sound
(and complete for interventional queries)

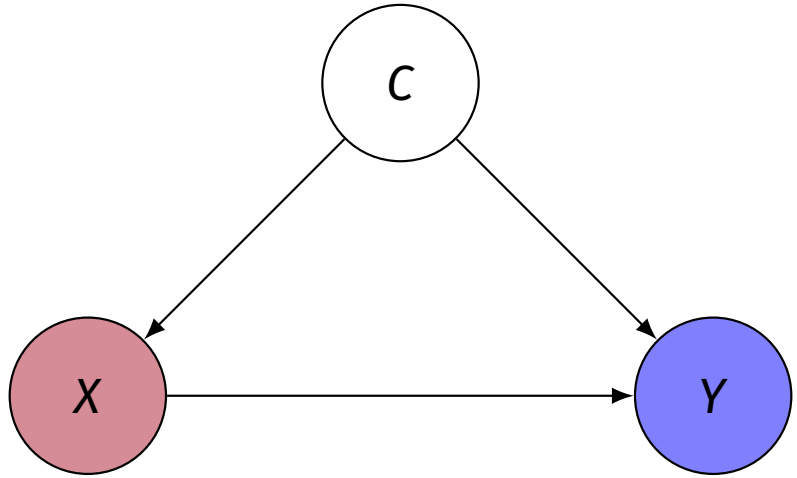
Rule 1 $\Pr(y \mid \mathrm{do}(x), z, w) = \Pr(y \mid \mathrm{do}(x), w)$
if $(Y \perp\!\!\!\perp_d Z \mid X, W)_{\mathcal{G}_{\overline{X}}}$

Rule 2 $\Pr(y \mid \mathrm{do}(x, z), w) = \Pr(y \mid \mathrm{do}((x)), z, w)$
if $(Y \perp\!\!\!\perp_d Z \mid X, W)_{\mathcal{G}_{\overline{X}\underline{Z}}}$

Rule 3 $\Pr(y \mid \mathrm{do}(x, z), w) = \Pr(y \mid \mathrm{do}(x), w)$
if $(Y \perp\!\!\!\perp_d Z \mid X, W)_{\mathcal{G}_{\overline{X}\overline{Z(W)}}}$

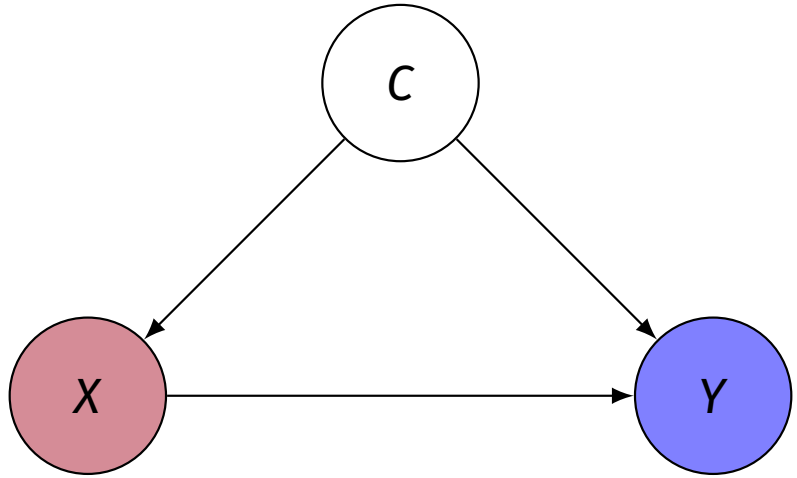
Pearl 1995

The backdoor: an application of the do-calculus



ATE: $\mathbb{E}(Y \mid \mathrm{do}(X = x_1)) - \mathbb{E}(Y \mid \mathrm{do}(X = x_0))$

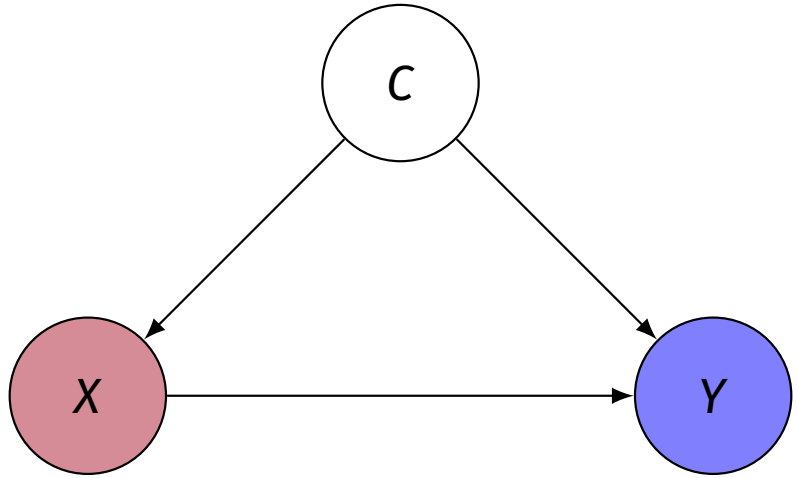
The backdoor: an application of the do-calculus



ATE: $\mathbb{E}(Y \mid \mathrm{do}(X = x_1)) - \mathbb{E}(Y \mid \mathrm{do}(X = x_0))$

$\Pr(Y \mid \mathrm{do}(X))$

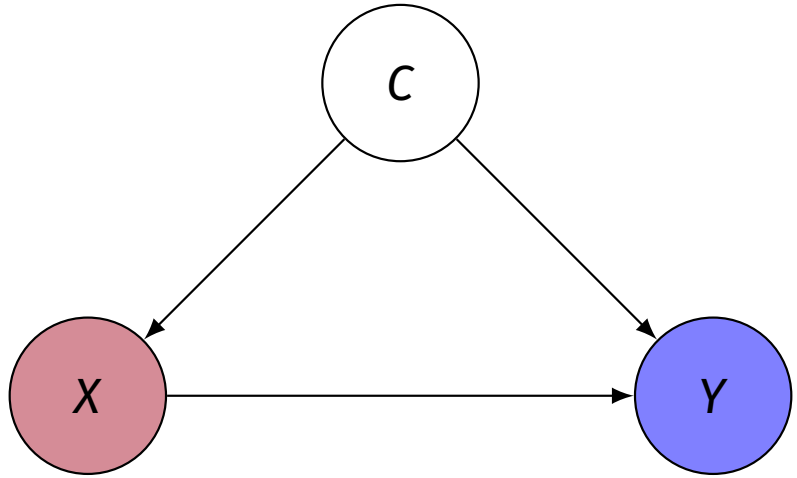
The backdoor: an application of the do-calculus



$$\text{ATE: } \mathbb{E}\left(Y \mid \text{do}\left(X = x_1\right)\right) - \mathbb{E}\left(Y \mid \text{do}\left(X = x_0\right)\right)$$

$$\Pr\left(Y \mid \text{do}\left(X\right)\right) = \sum_c \Pr\left(Y \mid \text{do}\left(X\right), C = c\right) \Pr\left(C = c \mid \text{do}\left(X\right)\right)$$

The backdoor: an application of the do-calculus



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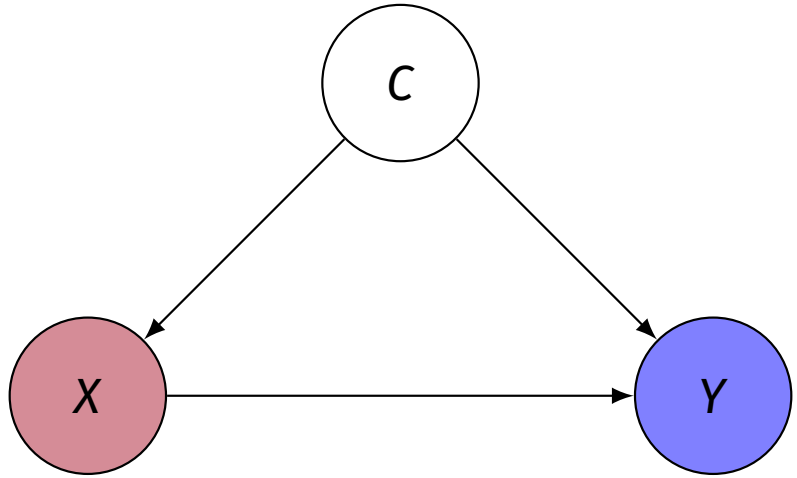
The backdoor: an application of the do-calculus



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PO-CALCULUS

Rule 1 $\Pr\left(\mathbb{Y}_{do(x)} \mid \mathbb{Z}_{do(x)}, \mathbb{W}_{do(x)}\right) = \Pr\left(\mathbb{Y}_{do(x)} \mid \mathbb{W}_{do(x)}\right)$

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Rule 2 $\Pr\left(\mathbb{Y}_{do(x,z)} \mid \mathbb{W}_{do(x,z)}\right) = \Pr\left(\mathbb{Y}_{do(x)} \mid \mathbb{Z}_{do(x)}, \mathbb{W}_{do(x)}\right)$

if $\left(\mathbb{Y}_{do(x,z)} \perp\!\!\!\perp_d \mathbb{Z}_{do(x,z)} \mid \mathbb{W}_{do(x,z)}\right)_{\mathcal{G}_{do(x,z)}}$

Rule 3 $\Pr\left(\mathbb{Y}_{do(x,z)} \mid \mathbb{W}_{do(x,z)}\right) = \Pr\left(\mathbb{Y}_{do(x)} \mid \mathbb{W}_{do(x)}\right)$

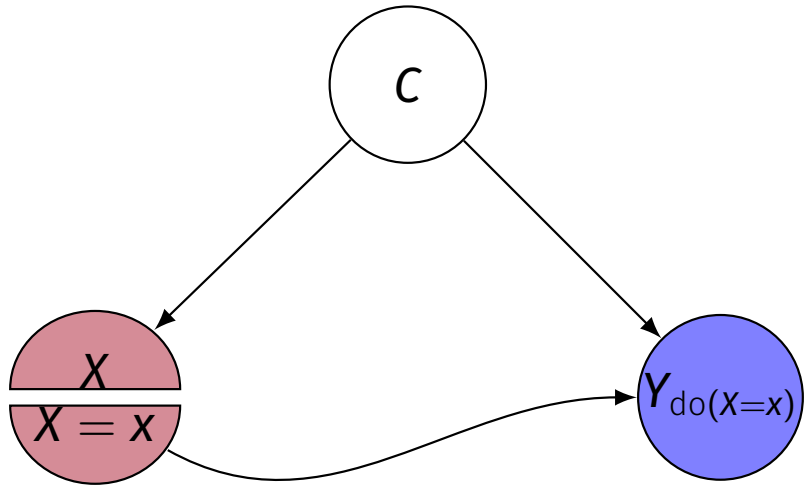
if $\left(\mathbb{Y}_{do(x,z_1)}, \mathbb{W}_{do(x,z_1)} \perp\!\!\!\perp_d \mathbb{Z}_1\right) \mathcal{G}_{do(x,z_1)}$

and $\left(\mathbb{Y}_{do(x,z_1)} \perp\!\!\!\perp_d \mathbb{Z}_{2do(x,z_1)} \mid \mathbb{W}_{do(x,z_1)}\right)_{\mathcal{G}_{do(x,z_1)}}$

where $Z_1 = Z \setminus An\left(W, \mathcal{G}_{do(x)}\right)$ and $Z_2 = Z \cap An\left(W, \mathcal{G}_{do(x)}\right)$

Malinsky, Shpitser, and Richardson 2019

The backdoor: an application of the po-calculus



$$\text{ATE: } \mathbb{E}\left(Y_{\text{do}(X=x_1)}\right) - \mathbb{E}\left(Y_{\text{do}(X=x_0)}\right)$$

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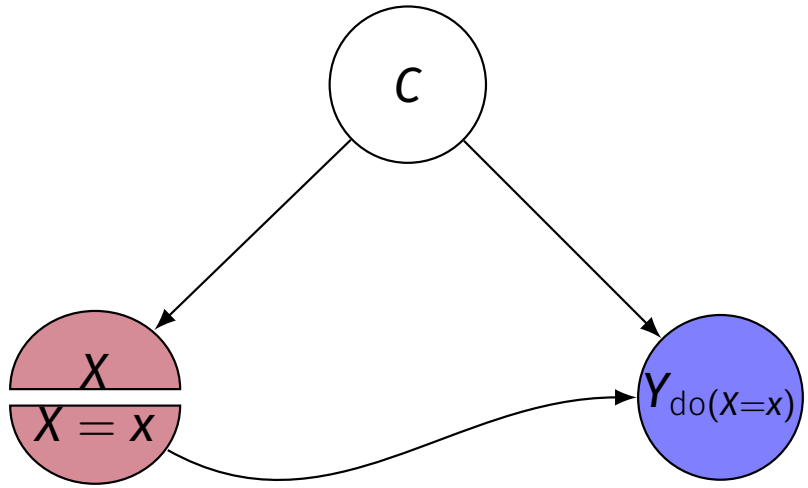

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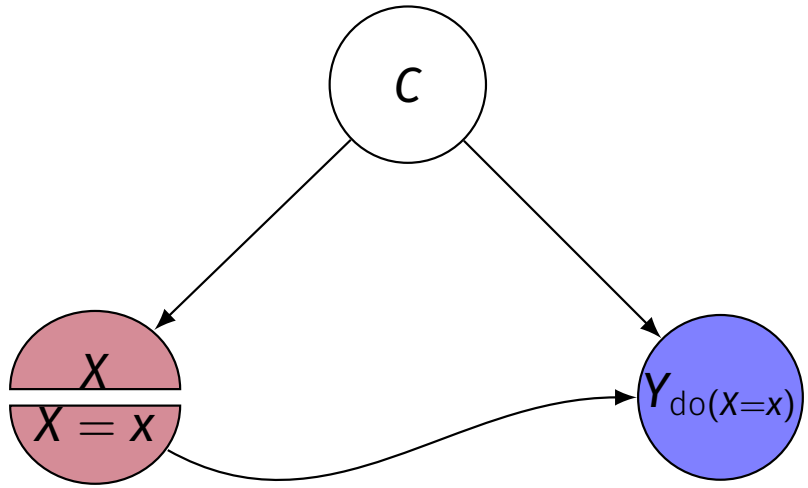
$$\Pr\left(Y_{\text{do}(X)}\right)$$

The backdoor: an application of the po-calculus



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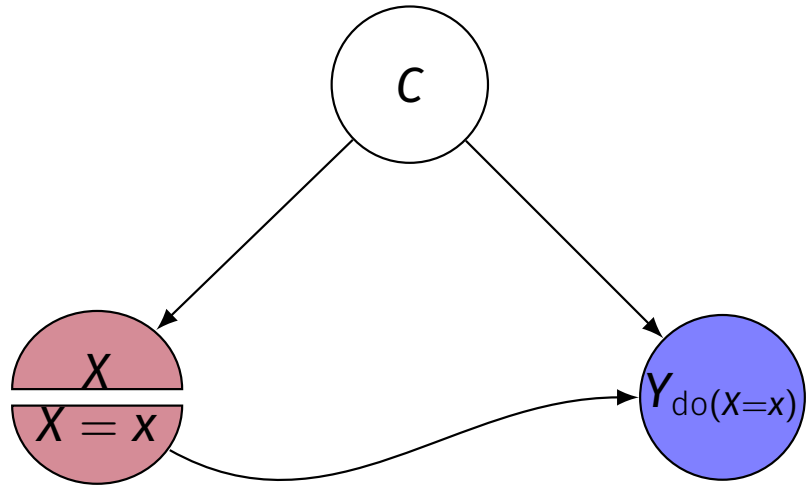

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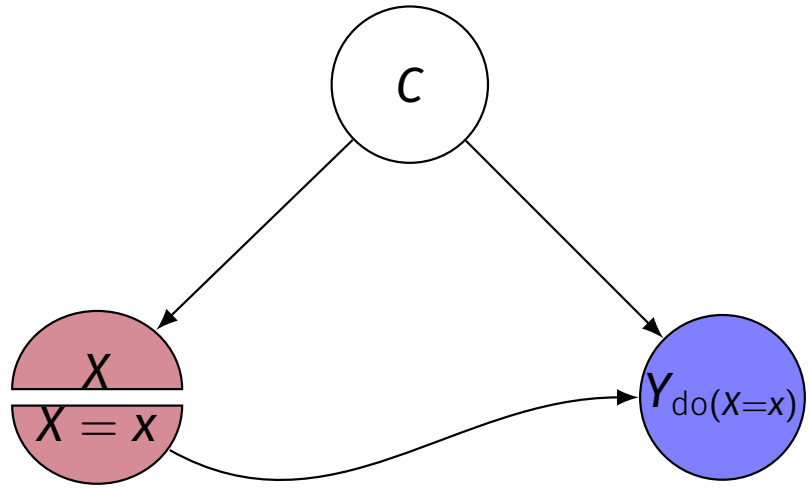

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Limitation of SWIGs

SWIGs: Single World Interventional Graphs

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Limitation of SWIGs

SWIGs: Single World Interventional Graphs

What is the effect of in the population that would die if they do not take it?

$$\mathbb{E}\left(Y_{do(X=1)} \mid Y_{do(X=0)} = 0\right)$$

Limitation of SWIGs

SWIGs: Single World Interventional Graphs

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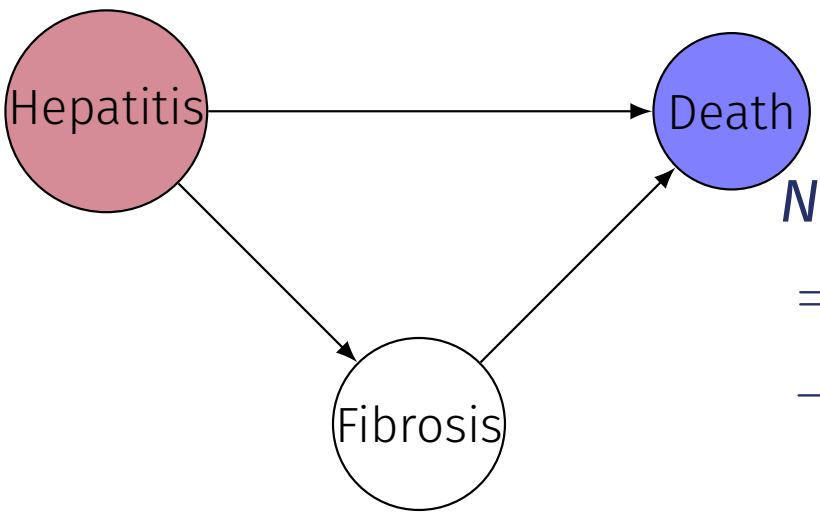
$$\mathbb{E}\left(Y_{\text{do}(X=1)} \mid Y_{\text{do}(X=0)} = 0\right)$$



$$\begin{aligned} &NDE(D, H_{0\to 1}, F) \\ &= \mathbb{E}\left(D \mid \text{do}\left(H = 1, F = f_{\text{do}(H=0)}\right)\right) \\ &- \mathbb{E}\left(D \mid \text{do}\left(H = 0, F = f_{\text{do}(H=0)}\right)\right) \end{aligned}$$

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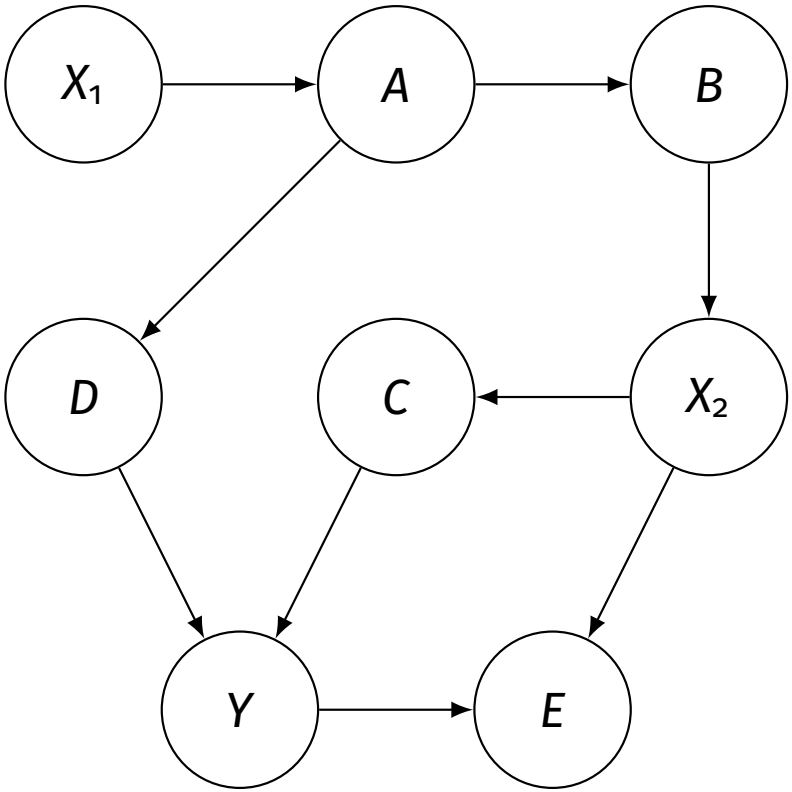
Ancestral Multi World Network (AMWN)

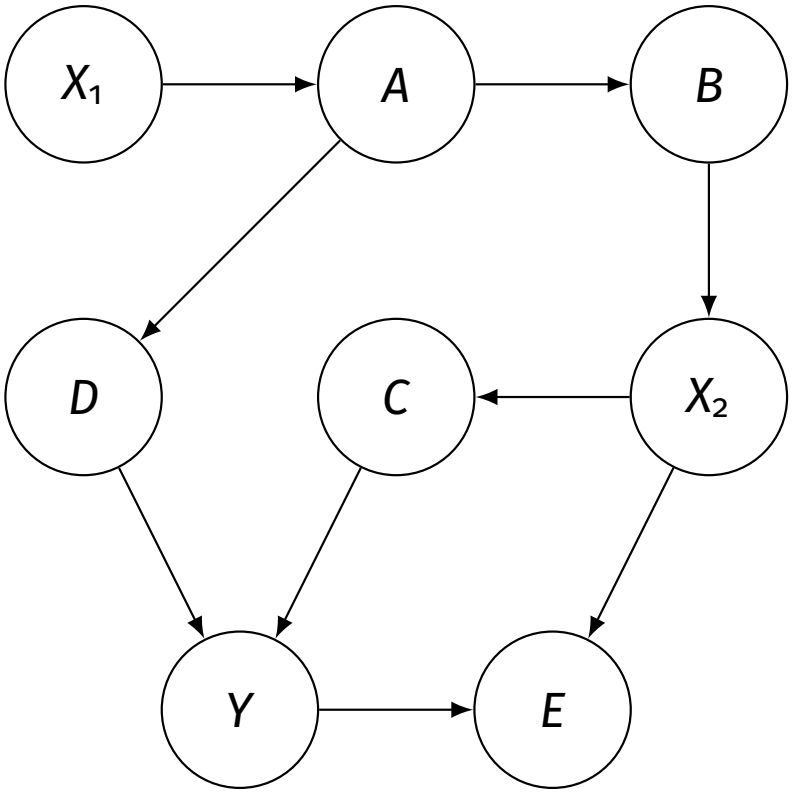
Ancestor of a Counterfactual Variable



$$An(\mathbb{Y}_{do(\mathbb{x})}) = \{W_{do(\mathbb{z})} \mid W \in An\left(Y, G_{\overline{\mathbb{X}}}\right) \setminus \mathbb{X}, \mathbb{z} = \mathbb{x} \cap An\left(W, G_{\overline{\mathbb{X}}}\right)\}$$

Correa, Lee, and Bareinboim 2021

Ancestor of a Counterfactual Variable



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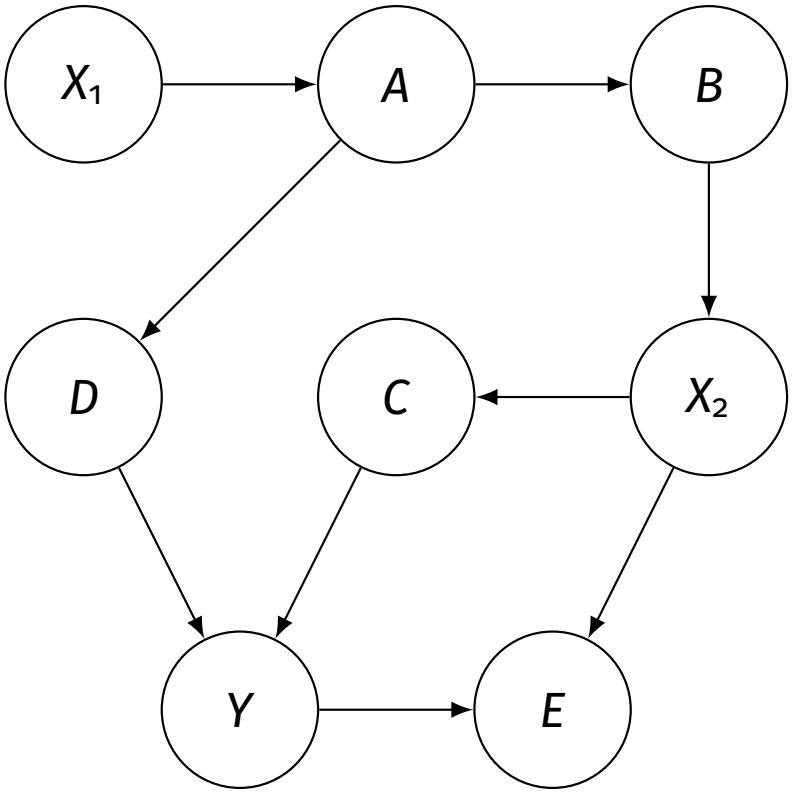
$An(Y_{do(X_1,X_2)})$?

Correa, Lee, and Bareinboim 2021

Ancestor of a Counterfactual Variable



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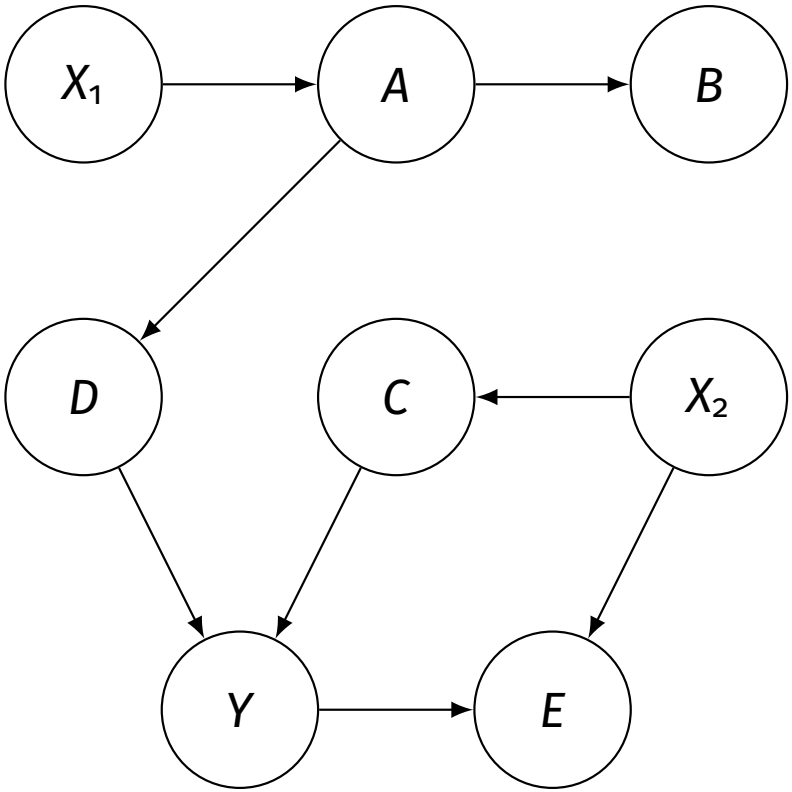
$An(Y_{do(X_1,X_2)})$?

- $G_{\overline{X_1,X_2}}$

Correa, Lee, and Bareinboim 2021

Ancestor of a Counterfactual Variable



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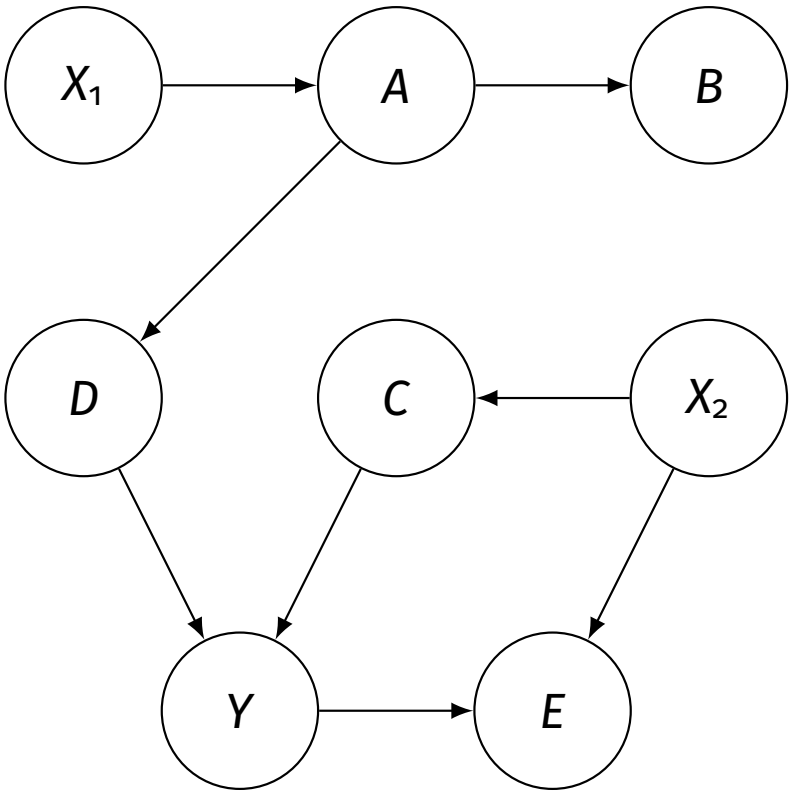
$An(Y_{do(X_1,X_2)})$?

- $G_{\overline{X_1,X_2}}$

Correa, Lee, and Bareinboim 2021

Ancestor of a Counterfactual Variable



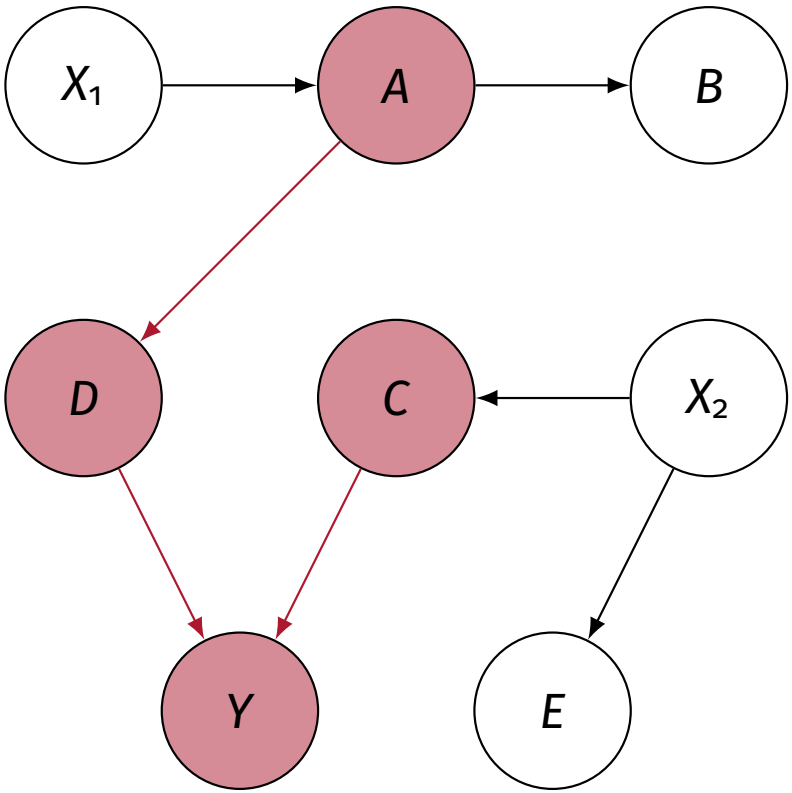
$$An(\mathbb{Y}_{do(\mathbb{x})}) = \{W_{do(\mathbb{z})} \mid W \in An\left(Y, G_{\overline{\mathbb{X}}}\right) \setminus \mathbb{X}, \mathbb{z} = \mathbb{x} \cap An\left(W, G_{\overline{\mathbb{X}}}\right)\}$$

$An(Y_{do(X_1,X_2)})$?

$\cdot\ G_{\overline{X_1,X_2}}$ $\cdot\ An\left(Y, G_{\overline{X_1,X_2}}\right) \setminus \{X_1, X_2\}$

Correa, Lee, and Bareinboim 2021

Ancestor of a Counterfactual Variable



$$An(\mathbb{Y}_{do(\mathbb{x})}) = \{W_{do(\mathbb{z})} \mid W \in An\left(Y, G_{\overline{\mathbb{X}}}\right) \setminus \mathbb{X}, \mathbb{z} = \mathbb{x} \cap An\left(W, G_{\overline{\mathbb{X}}}\right)\}$$

$An(Y_{do(X_1,X_2)})$?

$\cdot\ G_{\overline{X_1,X_2}}$ $\cdot\ An\left(Y, G_{\overline{X_1,X_2}}\right) \setminus \{X_1, X_2\}$

Correa, Lee, and Bareinboim 2021

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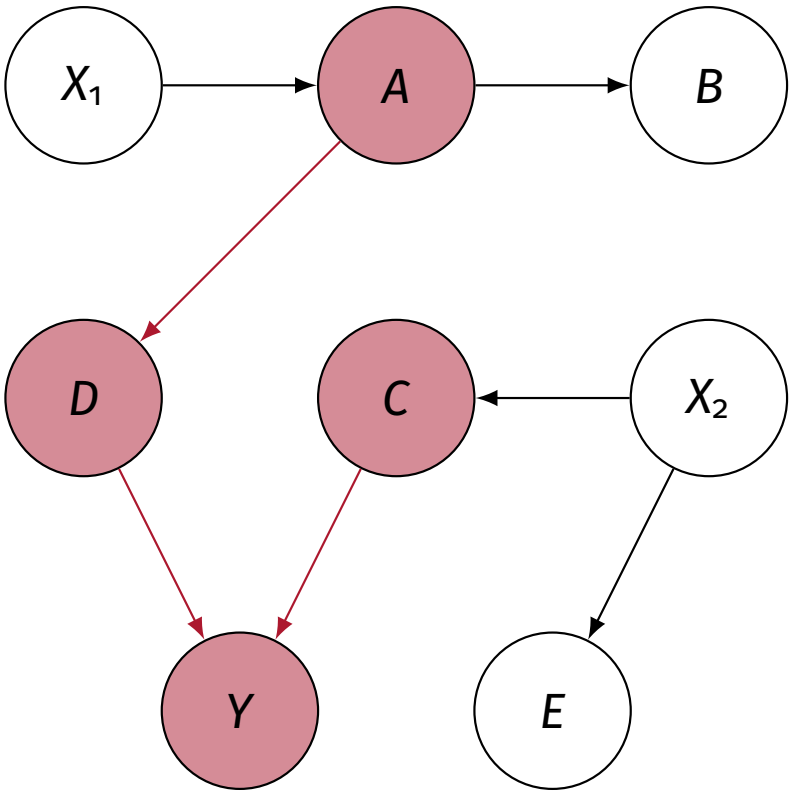

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Correa, Lee, and Bareinboim 2021

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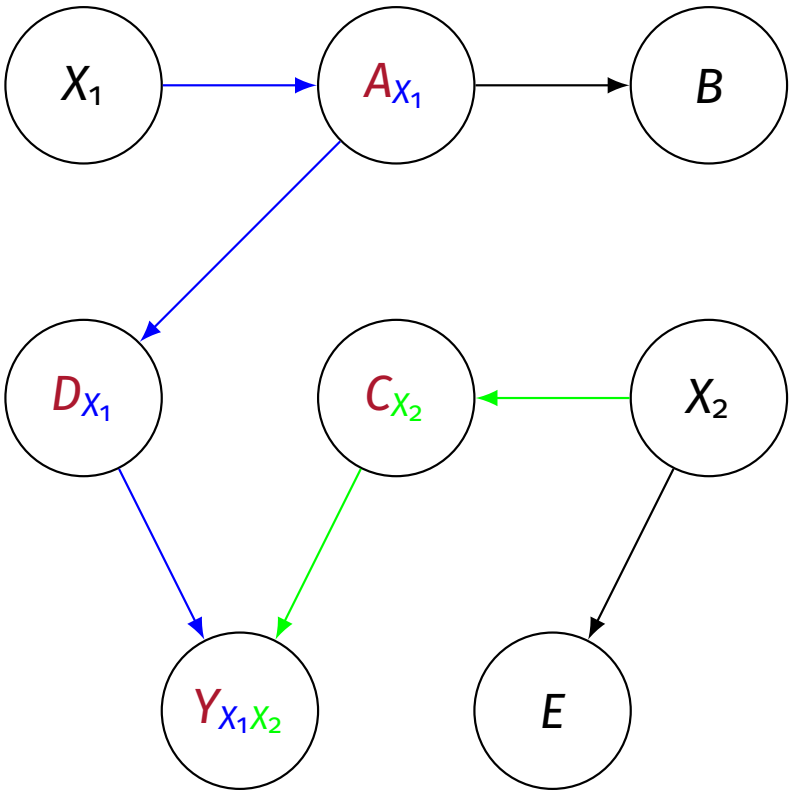

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Correa, Lee, and Bareinboim 2021

AMWN

For a query on counterfactual variables:

- Add all ancestors of the counterfactual variables
- Add edges witnessing the ancestrality
- Add the hidden variables
- Add hidden confounders between same variables

Correa and Bareinboim 2025

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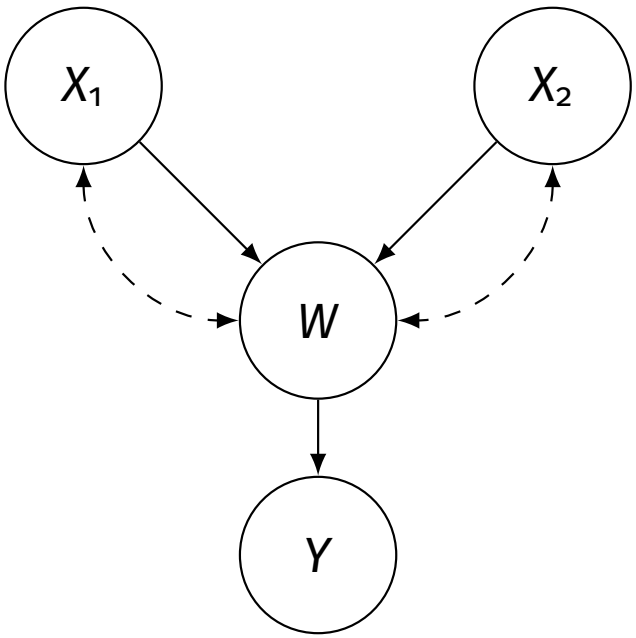

Correa and Bareinboim 2025

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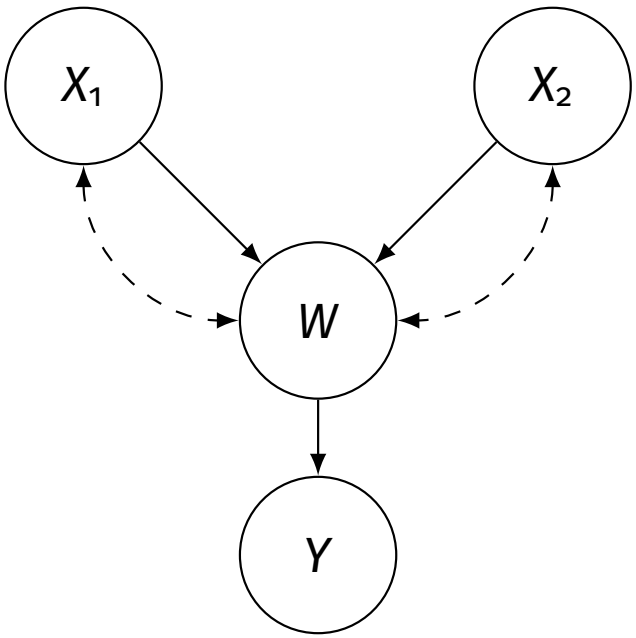

$$Y_{\text{do}(X_1)} \not\perp\!\!\!\perp_d X_2 \mid W_{\text{do}(X_1)}, W_{\text{do}(X_1,X_2)}$$

Correa and Bareinboim 2025

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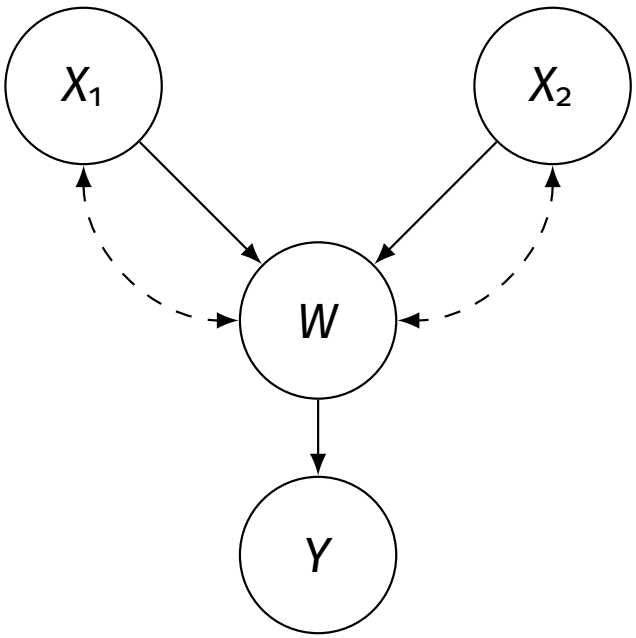

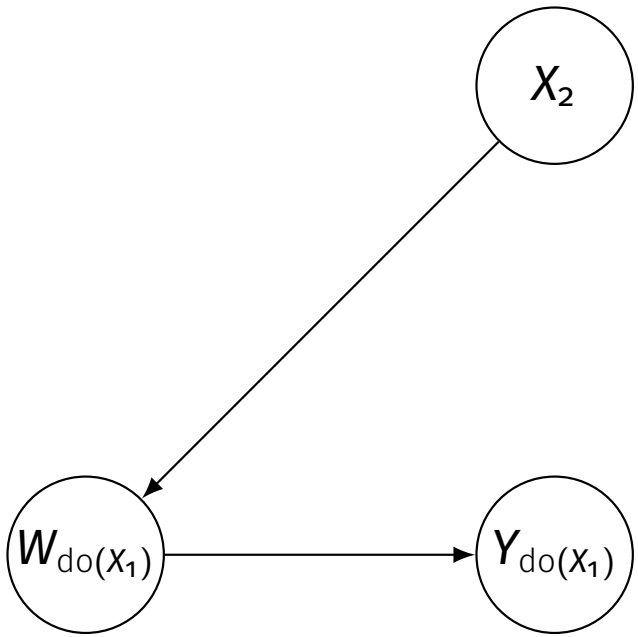

$$Y_{do(X_1)} \perp\!\!\!\perp_d X_2 \mid W_{do(X_1)}, W_{do(X_1,X_2)}$$

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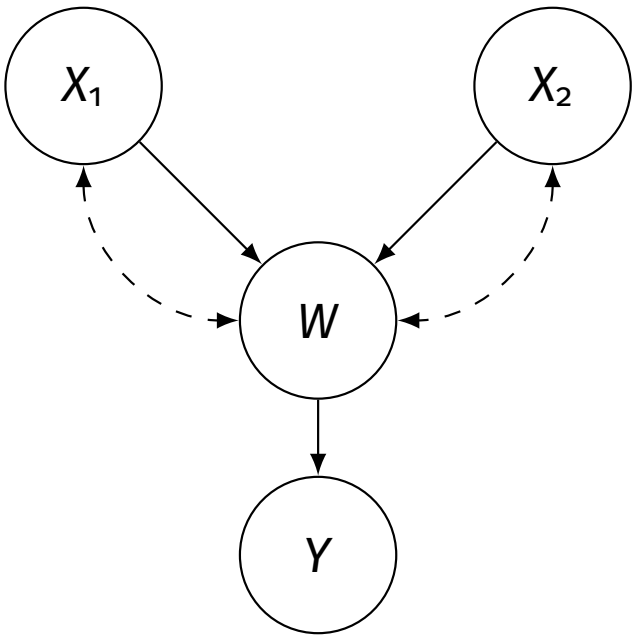

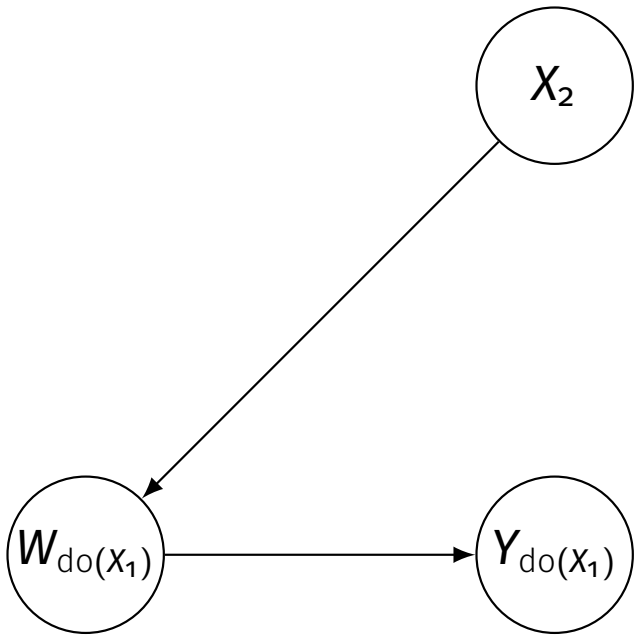

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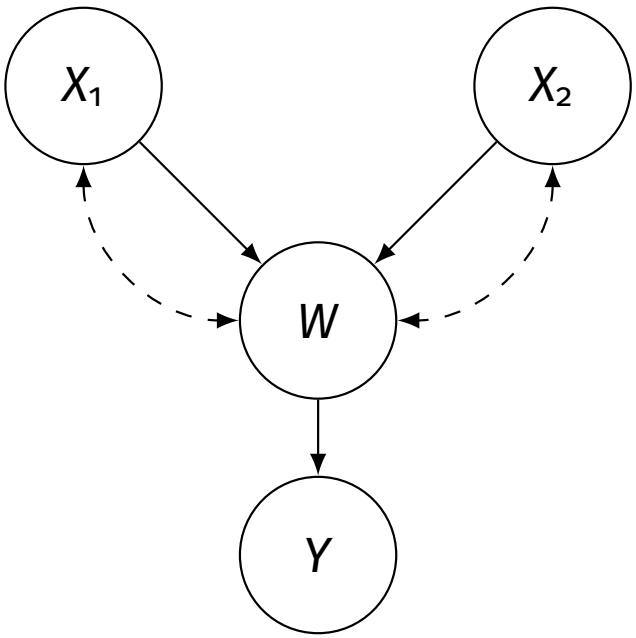

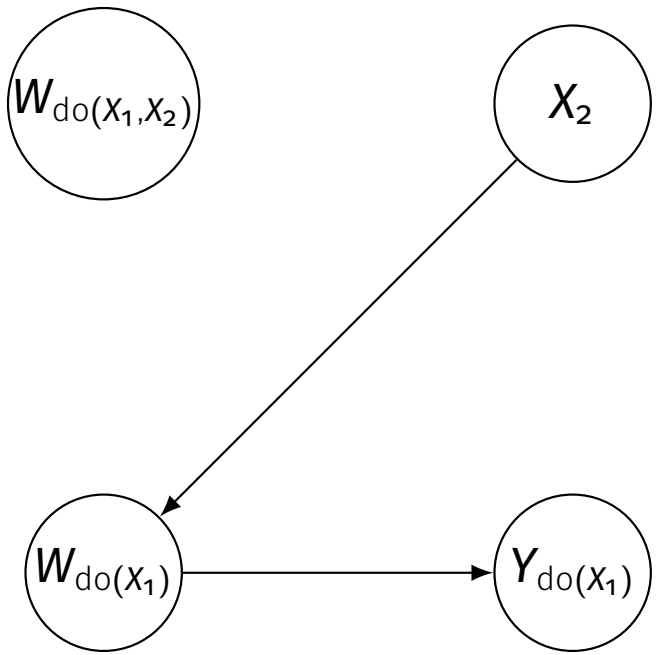

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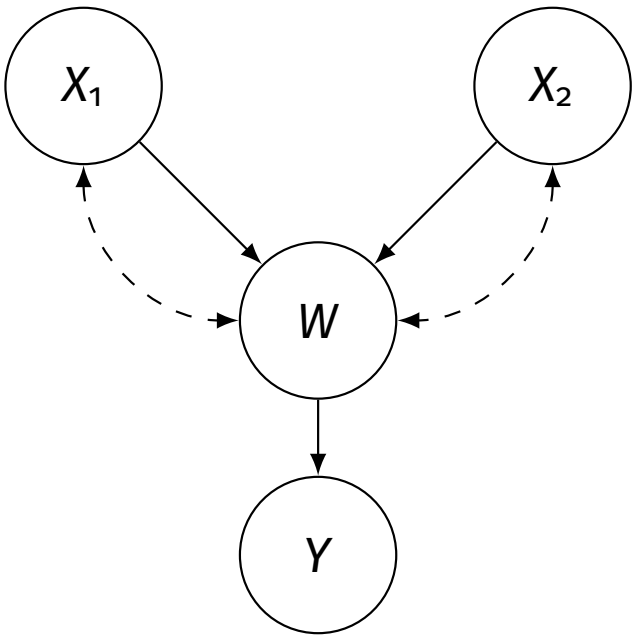

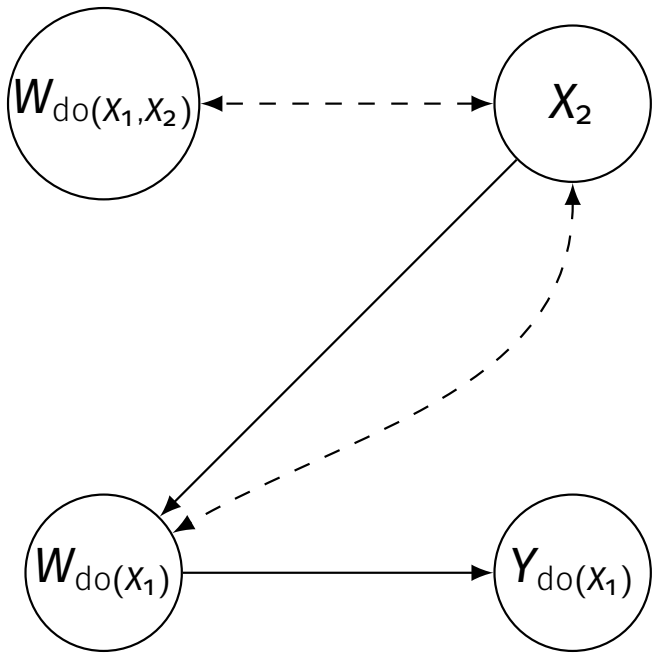

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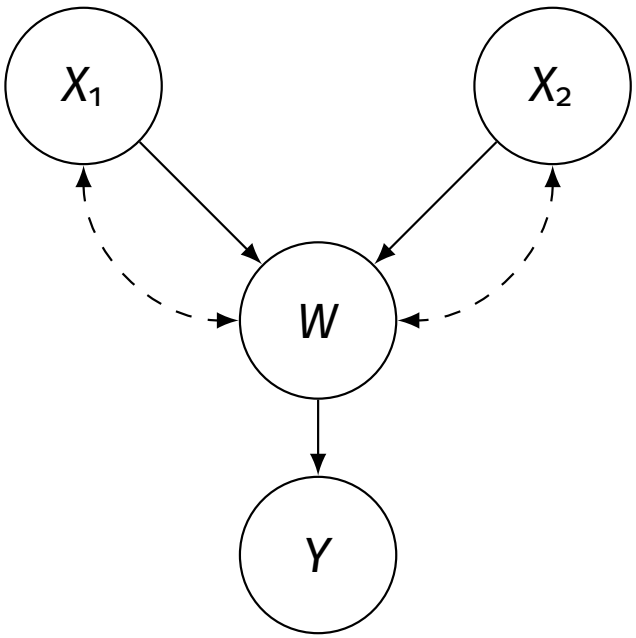

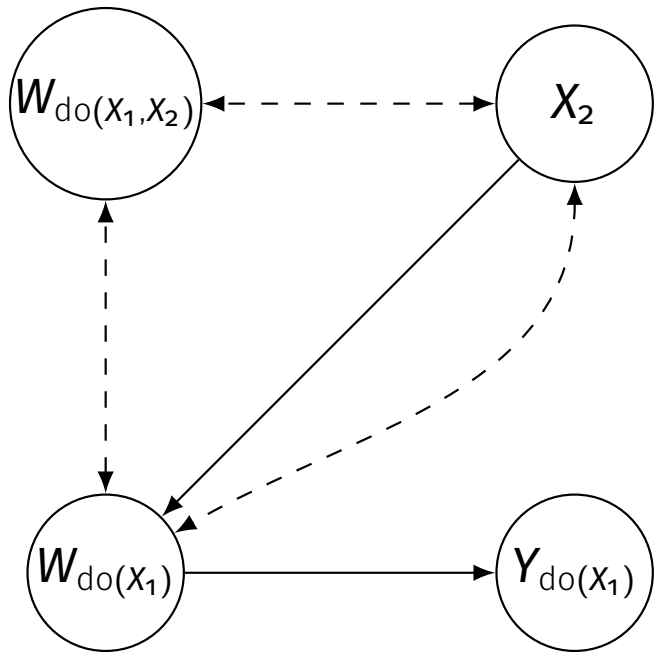

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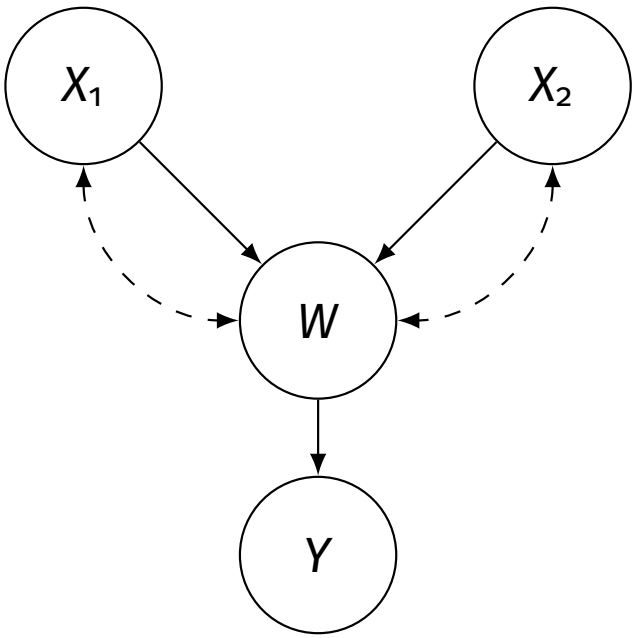
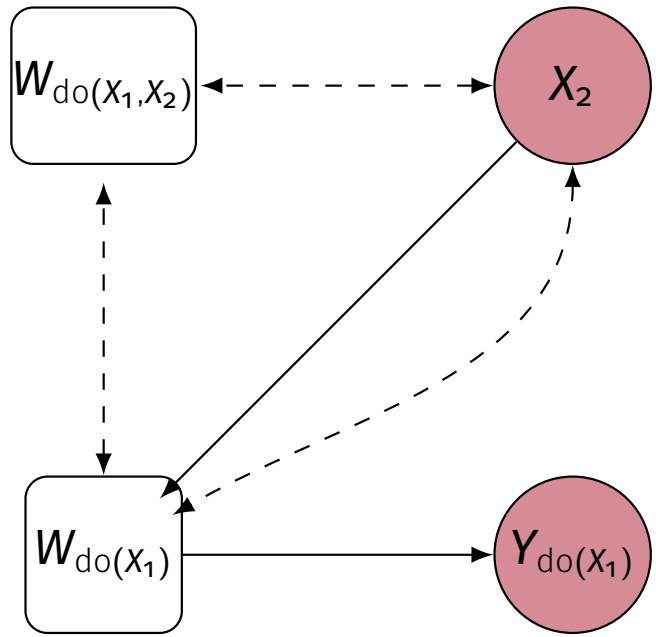




$$Y_{do(X_1)} \perp\!\!\!\perp_d X_2 \mid W_{do(X_1)}, W_{do(X_1,X_2)}$$

Correa and Bareinboim 2025

Counterfactual(Ctf)-Calculus

The ctf-calculus is sound and complete for counterfactual queries.

Rule 1 $\Pr(y_{T_* x}, x_{T_*}, w_*) = \Pr(Y_{T_*}, x_{T_*}, w_*)$

Rule 2 $\Pr(y_r \mid x_t, w_*) = \Pr(y_r \mid w_*)$

$$\text{if } (Y_r \perp\!\!\!\perp_d X_t \mid W_*)_{\mathcal{G}_A}$$

Rule 3 $\Pr(Y_{xz}, w_*) = \Pr(y_z, w_*)$

$$\text{if } X \cap An\left(Y, \mathcal{G}_{\overline{Z}}\right) = \emptyset$$

Correa and Bareinboim 2025

Counterfactual(Ctf)-Calculus

The ctf-calculus is sound and complete for counterfactual queries.

Rule 1 $\Pr\left(y_{T_* x}, x_{T_*}, w_*\right) = \Pr\left(Y_{T_*}, x_{T_*}, w_*\right)$

Rule 2 $\Pr\left(y_r \mid x_t, w_*\right) = \Pr\left(y_r \mid w_*\right)$

$$\text{if } \left(Y_r \perp\!\!\!\perp_d X_t \mid W_*\right)_{\mathcal{G}_A}$$

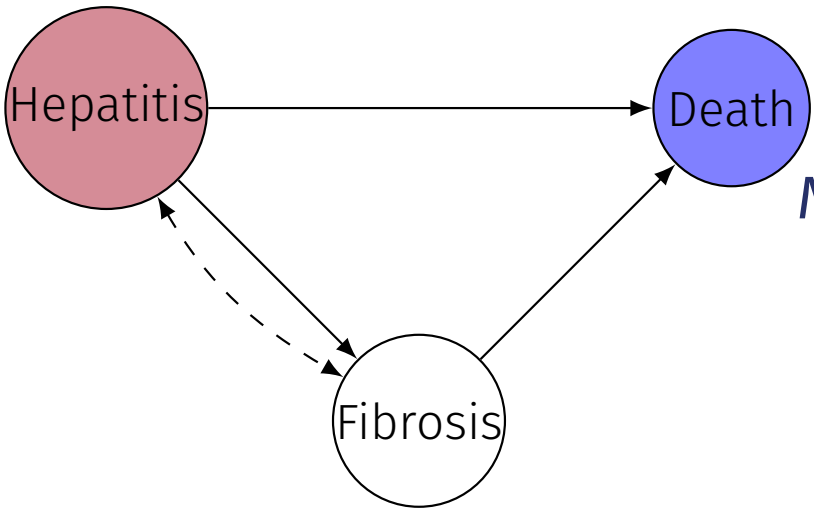
Rule 3 $\Pr\left(Y_{xz}, w_*\right) = \Pr\left(y_z, w_*\right)$

$$\text{if } X \cap An\left(Y, \mathcal{G}_{\overline{Z}}\right) = \emptyset$$

It transforms counterfactual queries into interventional queries. You might still need to use the do-calculus aftewards to transform the interventional queries into observational queries.

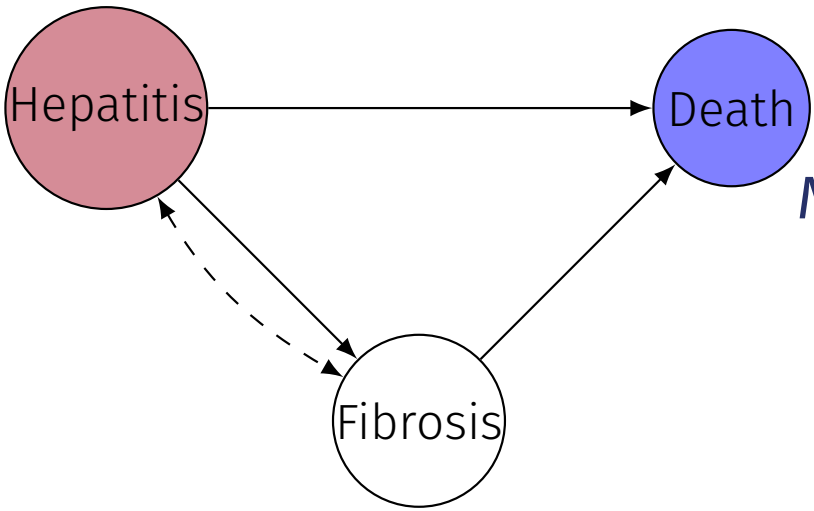
Correa and Bareinboim 2025

The Natural Direct Effect



$$NDE(D, H_{0\to 1}, F)$$
$$= \mathbb{E}\left(D \mid \text{do}\left(H = 1, F = f_{\text{do}(H=0)}\right)\right)$$
$$- \mathbb{E}\left(D \mid \text{do}\left(H = 0, F = f_{\text{do}(H=0)}\right)\right)$$

The Natural Direct Effect



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How do you identify
$\Pr\left(D \mid \text{do}\left(H = 1, F = f_{\text{do}(H=0)}\right)\right) = \Pr\left(d_{h' F_h}\right)$?

CTF-CALCULUS: EXAMPLE OF CALCULATIONS

$$\Pr\left(d_{h'F_h}\right)$$

CTF-CALCULUS: EXAMPLE OF CALCULATIONS

$$\Pr\left(d_{h'F_h}\right) = \sum_f \Pr\left(d_{h'F_h}, f\right)$$

CTF-CALCULUS: EXAMPLE OF CALCULATIONS

$$\Pr\left(d_{h'F_h}\right) = \sum_f \Pr\left(d_{h'F_h}, f\right)$$

$$(R1) = \sum_f \Pr\left(d_{h'f}, f_h\right)$$

CTF-CALCULUS: EXAMPLE OF CALCULATIONS

$$\begin{aligned}
\Pr\left(d_{h'F_h}\right) &= \sum_f \Pr\left(d_{h'F_h}, f\right) \\
(R1) &= \sum_f \Pr\left(d_{h'f}, f_h\right) \\
&= \sum_f \Pr\left(d_{h'f} \mid f_h\right) \Pr\left(f_h\right)
\end{aligned}$$

CTF-CALCULUS: EXAMPLE OF CALCULATIONS

$$\Pr\left(d_{h'F_h}\right) = \sum_f \Pr\left(d_{h'F_h}, f\right)$$

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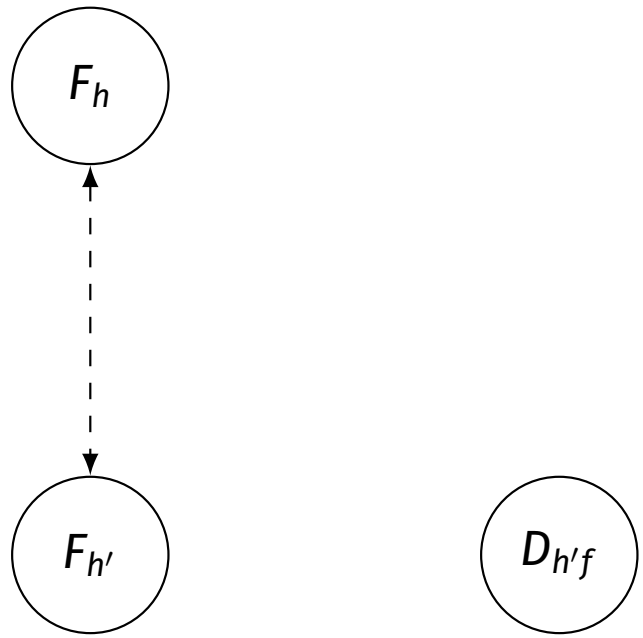
$$(R2) = \sum_f \Pr\left(d_{h'f} \mid f_{h'}\right) \Pr\left(f_h\right)$$



$$\left(D_{h'f} \perp\!\!\!\perp_d F_h, F'_h\right)_{\mathcal{G}_A}$$

CTF-CALCULUS: EXAMPLE OF CALCULATIONS

$$\Pr\left(d_{h'F_h}\right) = \sum_f \Pr\left(d_{h'F_h}, f\right)$$

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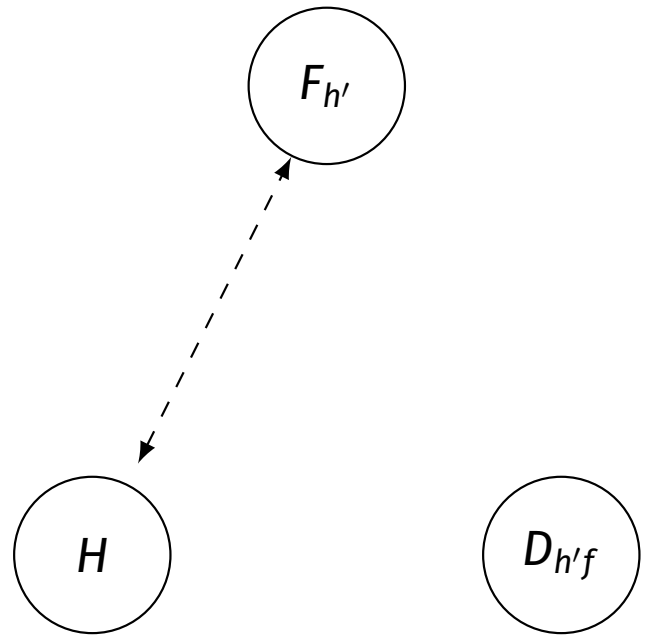

$$\left(D_{h'f} \perp\!\!\!\perp_d H \mid F_{h'}\right)_{\mathcal{G}_A}$$

CTF-CALCULUS: EXAMPLE OF CALCULATIONS

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\begin{aligned}
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(R2) &= \sum_f \Pr\left(d_{h'f} \mid f_{h'} h'\right) \Pr\left(f_h\right) \\
(R1) &= \sum_f \Pr\left(d_{h'f} \mid f, h'\right) \Pr\left(f_h\right)
\end{aligned}
$$

CTF-CALCULUS: EXAMPLE OF CALCULATIONS

$$
\begin{aligned}
\Pr\left(d_{h'F_h}\right) &= \sum_f \Pr\left(d_{h'F_h}, f\right) \\
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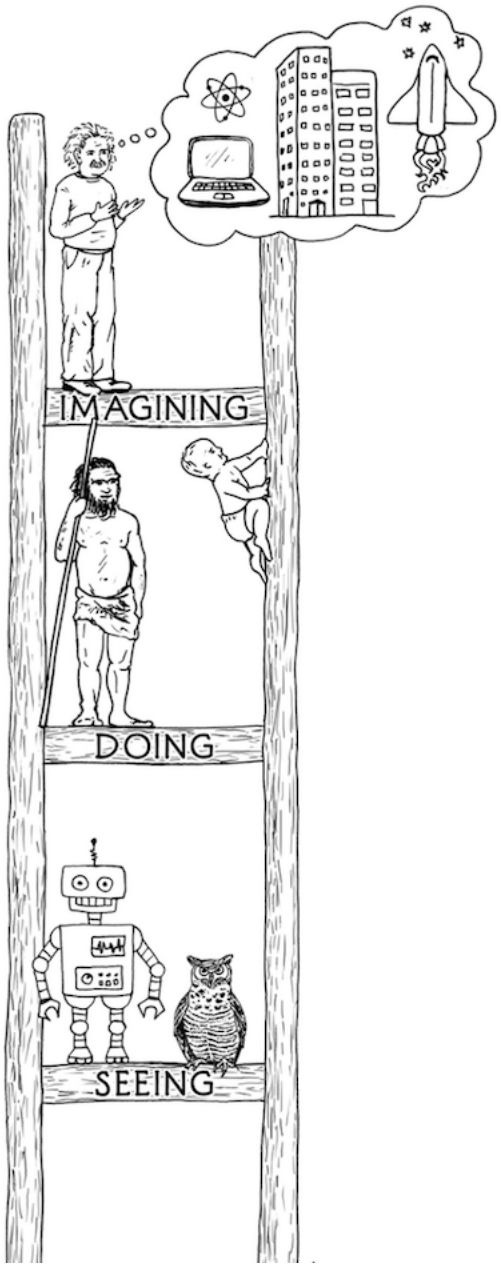
(R1) &= \sum_f \Pr\left(d \mid f, h'\right) \Pr\left(f_h\right)
\end{aligned}
$$

6

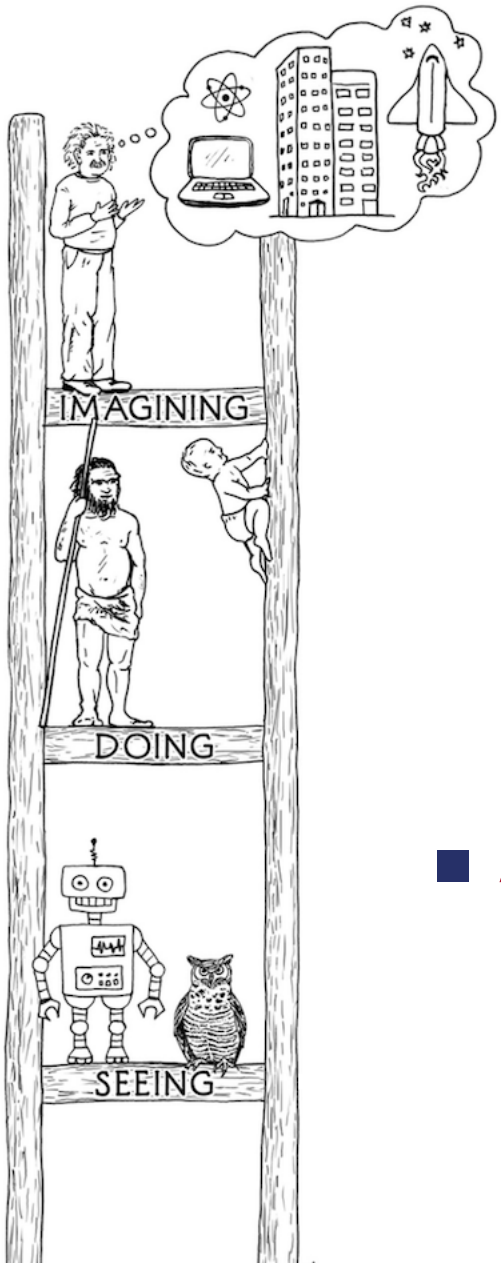
Conclusion

Conclusion



No assumptions

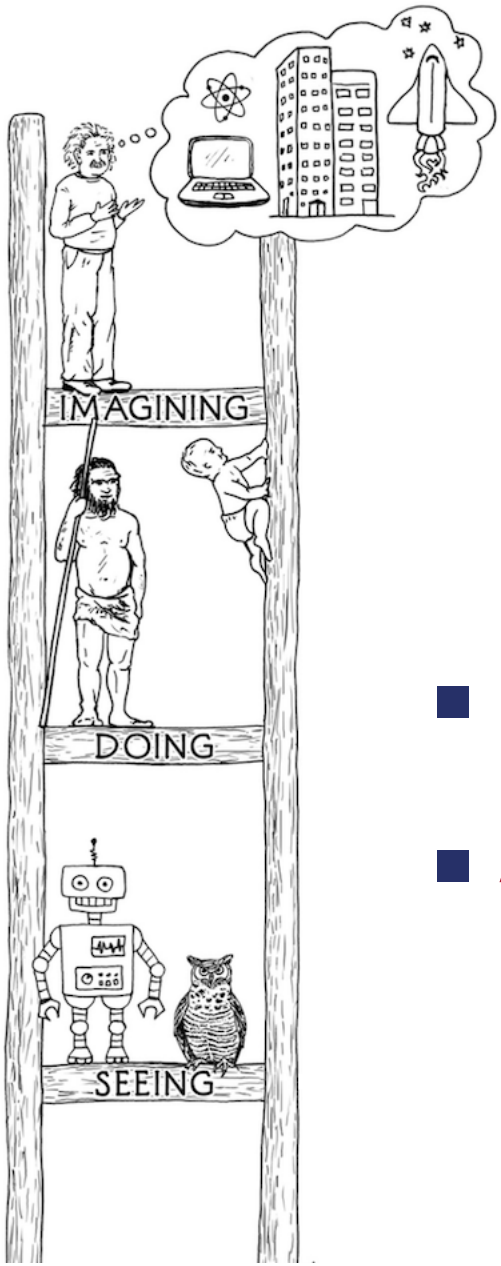
Conclusion



No assumptions

- Associations
  - observational data

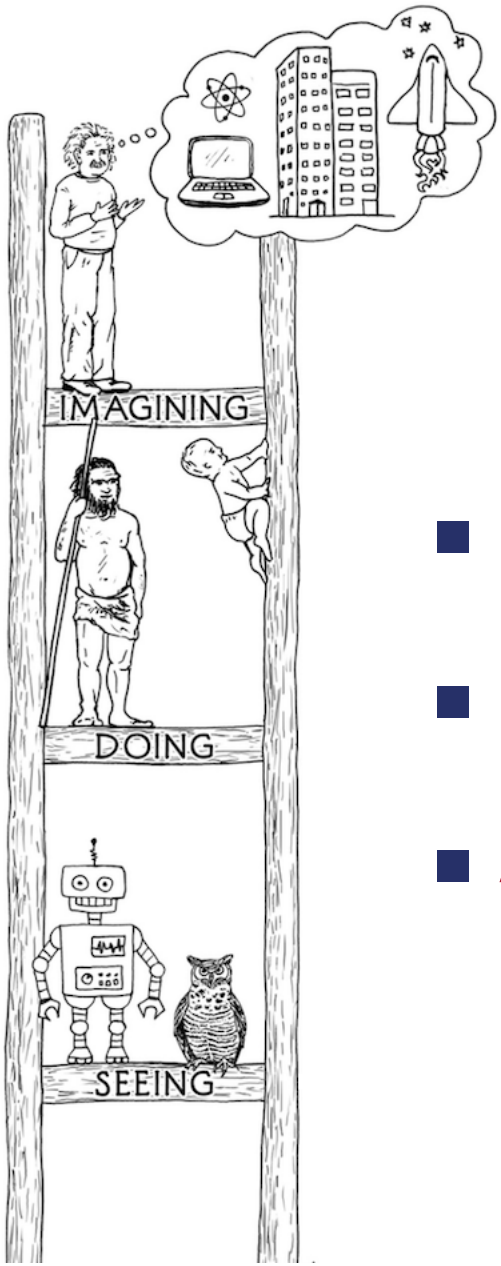
Conclusion



No assumptions

- Interventions
  - interventional data
- Associations
  - observational data

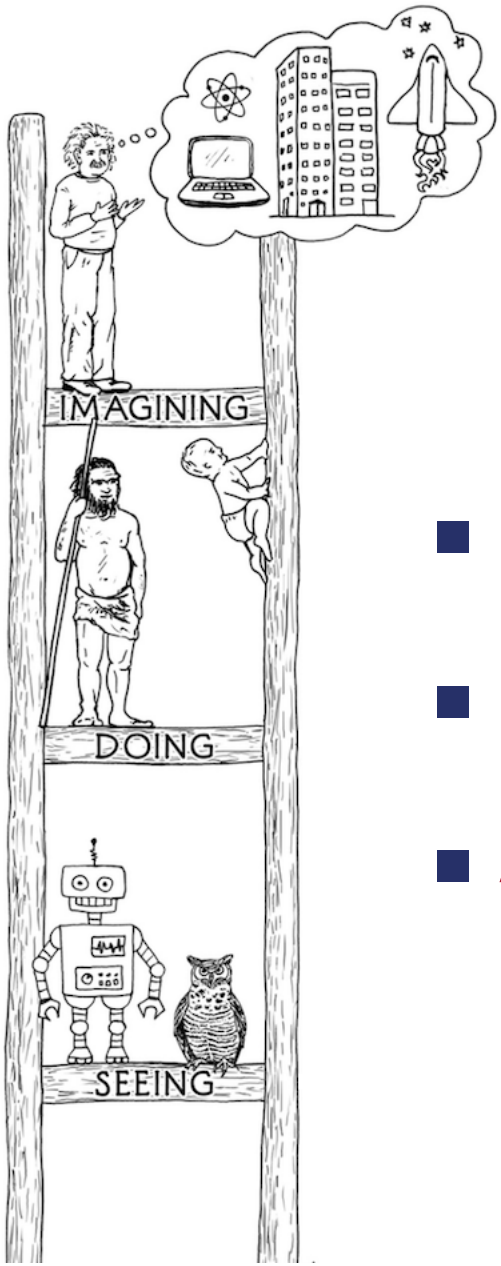
Conclusion



No assumptions

- Counterfactuals
  - Generating process
- Interventions
  - interventional data
- Associations
  - observational data

Conclusion



No assumptions

Assumptions: DAG

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  - Generating process
- Interventions
  - interventional data
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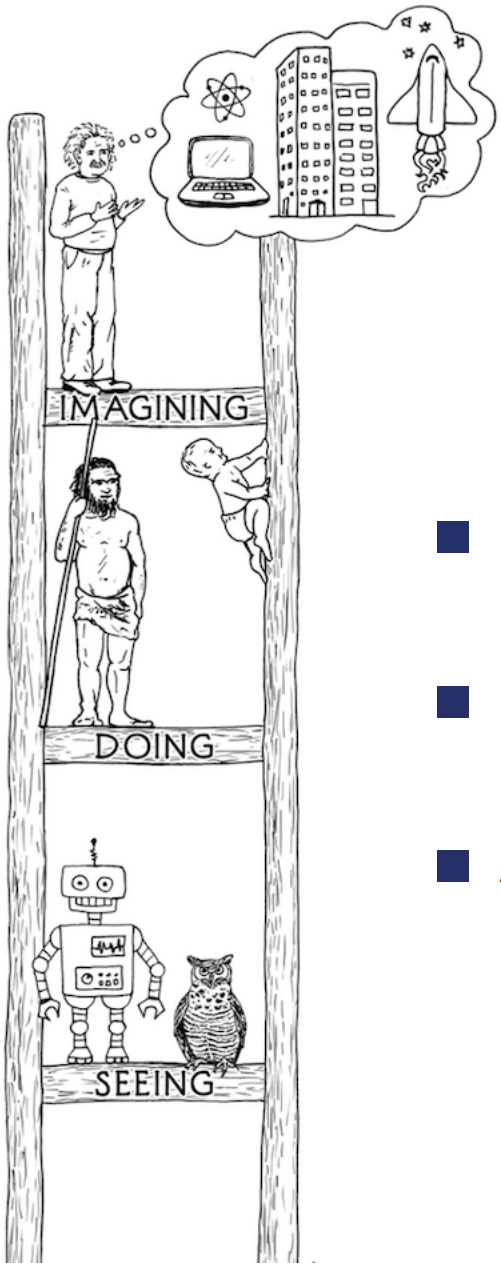
  - observational data

Conclusion



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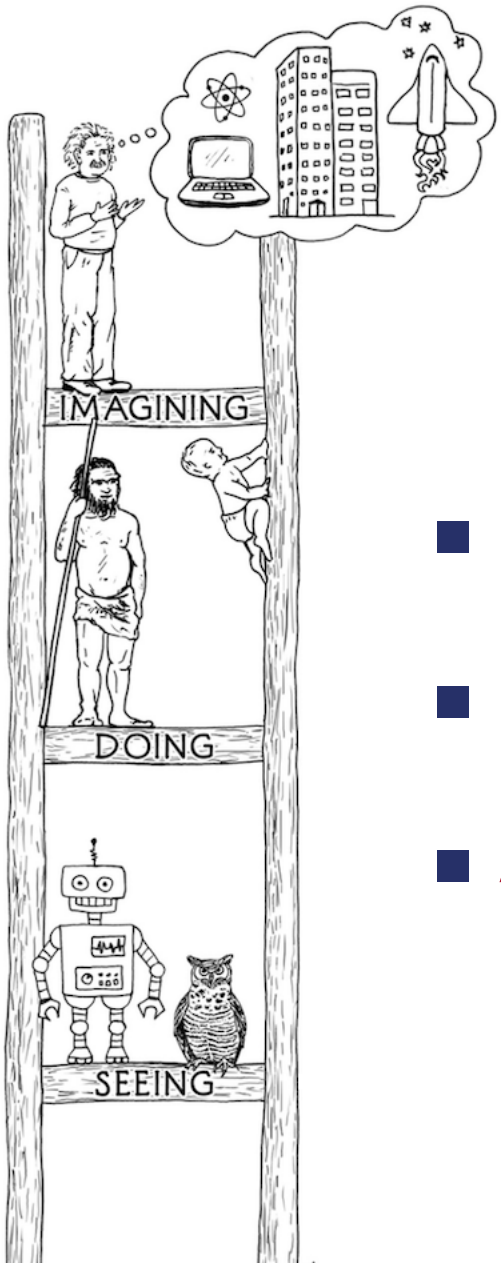
Assumptions: DAG

do-calculus

Conclusion



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Assumptions: DAG

ctf-calculus

do-calculus

THANK YOU FOR YOUR ATTENTION!

ANY QUESTIONS?

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